DUE DATE: OCT. 6, 2004

1. Let $\mathbf{F}(x, y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the function $\mathbf{F}(x, y) = (e^{x^2},\, y\sin(\pi x),\, xy)$, and $\mathbf{G} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the function $\mathbf{G}(u, v, w) = (\cos(uv), w - u^2)$.

(a) Compute $\mathbf{F}(1, 1)$, and let q be this point in $\mathbb{R}^3$.

(b) Write out the composite function $\mathbf{G} \circ \mathbf{F}$, and compute directly $\mathbf{D}(\mathbf{G} \circ \mathbf{F})(1, 1)$.

(c) Compute $\mathbf{DF}(1, 1)$ and $\mathbf{DG}(q)$.

(d) Compute the product $\mathbf{DG}(q)\,\mathbf{DF}(1, 1)$ and verify the chain rule in this case.

2. In a suitable coordinate system, the position of a satellite over top latitude α, longitude θ, and at a distance ρ from the center of the earth is given by

$$\begin{aligned} x &= \rho\sin(\alpha)\cos(\theta), \\ y &= \rho\sin(\alpha)\sin(\theta), \text{ and} \\ z &= \rho\cos(\alpha). \end{aligned}$$

There is a function f that we're interested in. At the point q in space described by $\rho = 2$, $\alpha = \pi/4$, and $\theta = \pi/3$ we know that

$$\begin{aligned} \frac{\partial f}{\partial x}(q) &= \sqrt{3}, \\ \frac{\partial f}{\partial y}(q) &= \sqrt{12}, \text{ and} \\ \frac{\partial f}{\partial z}(q) &= -1. \end{aligned}$$

(a) Compute $\dfrac{\partial f}{\partial \rho}(q)$, $\dfrac{\partial f}{\partial \alpha}(q)$, and $\dfrac{\partial f}{\partial \theta}(q)$.

(b) From this point, we wish to move in a direction so that f doesn't change. We also don't want to change ρ (it's more difficult to go up and down in orbit). With this in mind, what direction should we head in so that the instantaneous rate of change of f is zero?

3. A differential equation of the form $u_t = cu_{xx}$ where u is a function $u(x,t)$ of x and t, and c is a constant, is called a *diffusion equation.*

(a) Show that $u(x,t) = e^{ax+bt}$ (for a and b constants) satisfies the diffusion equation with $c = b/a^2$.

(b) Show that $u(x,t) = t^{-1/2}e^{-x^2/t}$ satisfies the diffusion equation with $c = 1/4$.

(c) Explain why there is no C^2 function $f(x,y)$ with $f_x(x,y) = e^x + xy$ and $f_y(x,y) = e^x + xy$. (This question has nothing to do with parts (a) and (b)).

4. Describe the curve which is the intersection of $x + z = 1$ and $x^2 + y^2 = 1$. Find a parameterization of this curve.

5.

(a) Describe and sketch the graph $z = \dfrac{1}{x^2 + y^2}$.

(b) Show that the parameterization $(x(t),\, y(t),\, z(t)) = (e^t \cos(t),\, e^t \sin(t),\, e^{-2t})$. lies on the graph from part (a).

(c) Describe what this curve does, and sketch it on the graph from part (a).