DUE DATE: OCT. 13, 2004

1. *Spirals*

   (a) For the parameterized curve given by $x(t) = e^t \cos(t)$ and $y(t) = e^t \sin(t)$, prove that at any point the vector joining that point to the origin meets the tangent line at that in a constant angle (i.e., an angle independent of t). Also find the angle.

   (b) Find the length of the above curve between the times $t = 0$ and $t = 5$.

   (c) (BONUS QUESTION) Suppose that we have a parameterization of a curve of the form $x(t) = r(t)\cos(t)$ and $y(t) = r(t)\sin(t)$, with the property similar to the property in part (a): for any point of the curve the vector joining the origin to that point meets the tangent line to that point at a constant angle θ. Show that $r(t)$ must be of the form $r(t) = e^{kt}$, and find the constant k in terms of θ.

2. *Velocity and acceleration*

   (a) If $\vec{u}(t) = (u_1(t), u_2(t))$ is a vector depending on t, compute a formula for the derivative $\frac{d}{dt}\|u(t)\|^2$ of the square of the length of this vector.

   (b) If $x(t)$ and $y(t)$ is a parameterized curve in $\mathbb{R}^2$, show that the speed of this parameterization is constant if and only if the acceleration is always perpendicular to the velocity vector.

   (c) If an object (like a planet) orbits around a more massive object (like the sun) the orbit will be an ellipse with the massive object at one of the two foci of the ellipse.

   The parameterization $x(t) = 2\cos(t)$ and $y(t) = \sin(t)$ is a parameterization of the ellipse $\frac{x^2}{4} + y^2 = 1$, which has foci at the points $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$. Could this parameterization be a parameterization of an object in orbit? Explain why or why not.

3. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function $f(x, y, z) = xy + z^2$ and C the parameterized curve given by $x(t) = 3t^2$, $y(t) = t\sin(t)$, and $z(t) = e^{2t}$.

   (a) Let p be the point on C corresponding to $t = \pi$, and $\vec{v}$ the velocity vector at that point. Find p and $\vec{v}$.

   (b) Find the gradient ∇f at the point p. At the point p, what is the instantaneous rate of change of f if we go in the direction (and with the speed) given by $\vec{v}$?

(c) Write out the composite function $f(x(t), y(t), z(t))$ and compute its derivative when $t = \pi$.

(d) The answers to (b) and (c) are of course the same. Explain why this is a consequence of the chain rule.

4. *Steepest descent*
Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function $f(x, y) = x^2 + \frac{1}{2}y^2$.

(a) Compute the gradient $\nabla f(x, y)$.

(b) If $(x(t),\ y(t))$ is a parameterized curve, what is the gradient at the point $(x(t), y(t))$?

(c) At $t = 0$ I'm at the point $(1, 1)$ and want to always walk so that the graph of f *decreases* the fastest. I also want my instantaneous speed at time t to be the same as the length of the gradient vector where I'm at. If $x(t)$ and $y(t)$ is the parameterization of my path, how do I express mathematically (in terms of $x(t)$ and $y(t)$) these conditions?

(d) Find the parameterized curve $(x(t),\ y(t))$ satisfying these conditions. (This will involve solving two simple differential equation).

(e) What curve in the plane does this parameterization trace out?

5. For the following vector fields, identify those which are conservative, and those which are not conservative. For those which are conservative, find the potential function f. For those which are not conservative, explain how you know this.

(a) $\mathbf{F}(x, y, z) = (y^2,\ 2xy + y^2,\ 2yz)$.

(b) $\mathbf{F}(x, y, z) = (x^2 + y^2,\ 2x^3 + 1,\ z^2)$.

(c) $\mathbf{F}(x, y, z) = (\cos(y) - z\cos(x),\ -x\sin(y),\ -\sin(x))$.

(d) $\mathbf{F}(x, y, z) = (\sin(z) - y\sin(x),\ \cos(x),\ x\sin(z))$.