DUE DATE: NOV. 3, 2004

1. Which of the following sets are connected? Which are simply connected?

   (a) $\mathbb{R}^2$ with the circle $x^2 + y^2 = 1$ removed.

   (b) $\mathbb{R}^3$ with the circle $x^2 + y^2 = 1$, $z = 0$ removed.

   (c) The set $\{(x, y) \mid 1 < x^2 + y^2 < 2\}$ in $\mathbb{R}^2$.

   (d) $\mathbb{R}^3$ with the helix $(\cos(t), \sin(t), t)$, $t \in [0, \pi]$ removed.

   (e) The set $\{(x, y) \mid x^2 - y^2 < 0\}$ in $\mathbb{R}^2$.

2. Here are three curves connecting the point $(1, 0, 0)$ to the point $(-1, 0, 0)$ in $\mathbb{R}^3$:

   $\mathbf{c}_1$: The half-circle $(\cos(t), \sin(t), 0)$, $t \in [0, \pi]$.
   $\mathbf{c}_2$: The segment $(-t, t^2 - 1, 1 - t^2)$ of a parabola, $t \in [-1, 1]$.
   $\mathbf{c}_3$: The straight line $(-t, 0, 0)$, $t \in [-1, 1]$.

   (a) For $\mathbf{F} = (-y, x, z)$, compute $\int_{\mathbf{c}_1} \mathbf{F} \cdot ds$, $\int_{\mathbf{c}_2} \mathbf{F} \cdot ds$, and $\int_{\mathbf{c}_3} \mathbf{F} \cdot ds$.

   (b) For $\mathbf{G} = (e^{yz}, xz\, e^{yz}, xy\, e^{yz})$, compute $\int_{\mathbf{c}_1} \mathbf{G} \cdot ds$, $\int_{\mathbf{c}_2} \mathbf{G} \cdot ds$, and $\int_{\mathbf{c}_3} \mathbf{G} \cdot ds$.

   (c) Is $\mathbf{F}$ a conservative vector field? Is $\mathbf{G}$?

3. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y) = \left(\frac{y}{\sqrt{x^2 + y^2}}, \frac{-x}{\sqrt{x^2 + y^2}} \right)$$

and $\mathbf{c}$ the unit circle, oriented counterclockwise.

   (a) What is the domain of definition of the vector field $\mathbf{F}$? Is it simply connected?

   (b) Compute Curl($\mathbf{F}$) (the "$\mathbb{R}^2$" curl, which is a function, and not a vector field).

   (c) Compute $\int_{\mathbf{c}} \mathbf{F} \cdot ds$.

   (d) If $\mathbf{G}$ is a vector field, and $\mathbf{G} = \nabla g$ for some function g, what would $\int_{\mathbf{c}} \mathbf{G} \cdot ds$ have to be? (HINT: Think of $\mathbf{c}$ as a curve whose ending point is the same as its starting point).

   (e) Explain how you know that $\mathbf{F}$ cannot be the gradient of any function, even though by a local calculation (the curl) it looks like it should be.

4. Let f be the function $f(x,y) = x^2y$, and

$\mathbf{c}_1$: The half circle $(\sqrt{2}\cos(t),\ \sqrt{2}\sin(t))$, $t \in [-3\pi/4,\ \pi/4]$.
$\mathbf{c}_2$: The half circle $(\sqrt{2}\cos(t),\ -\sqrt{2}\sin(t))$, $t \in [3\pi/4,\ 7\pi/4]$.
$\mathbf{c}_3$: The straight line (t,t) $t \in [-1,1]$.

All three curves connect the point $(-1,-1)$ to the point $(1,1)$.

(a) compute $f(1,1) - f(-1,-1)$

(b) Let $\mathbf{F} = \nabla f$. Compute $\mathbf{F}$.

(c) Compute $\int_{\mathbf{c}_1} \mathbf{F} \cdot ds$, $\int_{\mathbf{c}_2} \mathbf{F} \cdot ds$, and $\int_{\mathbf{c}_3} \mathbf{F} \cdot ds$.

(d) Explain the connection between (a) and (c).