The Three Wise Men: Green, Stokes, and Gauss

1. The three theorems we will discuss are very beautiful. Sorry, given my occupation, I had to say that. On a very practical level, however, they are also very useful. In some cases they will simplify a nasty looking integral to one that you can integrate with relative ease. It may seem strange that the simplification involves adding an extra integration (i.e. Green is line to surface in the two dimensional (planar) case, Stokes is line to surface in the three dimensional case, and Gauss is surface to volume in the three dimensions), but the way the integrand is simplified i.e. by taking the curl or divergence, more than makes up for it.

2.

   Theorem 1. *Greens Theorem*

   Let D be a region in $\mathbb{R}^2$:

   1) D is composed of elementary regions

   2) The positively oriented curve $c = \partial D$ completely encloses D

   3) The vector field F is C^1 over D

   then:

$$\int_c F \cdot ds = \int_c F \cdot T\, dt = \int\int_D \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dA$$

   Examples:

   (a) $\int_{\mathbf{C}} xy\, dx + x^2y^3\, dy$ where $\mathbf{C}$ is triangle with $(0,0), (0,1), (0,2)$ vertices. This line integral can be evaluated normally, with $\mathbf{C}$ broken into three curves - one for the horizontal line from $(0,0)$ to $(1,0)$, the vertical line from $(1,0)$ to $(1,2)$, and the line from $(1,2)$ to $(0,0)$, i.e. $y = 2x$ or we can do it "all at once" using Green's Theorem. We integrate over the area of the triangle, so, with $P(x,y) = xy$ and $Q(x,y) = x^2y^3$, we have:

$$\begin{aligned}
\oint_{\mathbf{C}} xy\, dx + x^2y^3\, dy &= \int_{x=0}^{1} \int_{y=0}^{2x} 2xy^3 - x\, dy\, dx \\
&= \int_{x=0}^{1} |_{y=0}^{y=2x} \frac{xy^4}{2} - xy\, dx \\
&= \int_{x=0}^{1} 8x^5 - 2x^2\, dx \\
&= |_{x=0}^{1} \frac{4x^6}{3} - \frac{2x^3}{3} = \frac{2}{3}
\end{aligned}$$

   Remember this works because $\mathbf{F} = (xy, x^2y^3)$ is C^1 over the area of the triangle.

(b) $\int_{\mathbf{C}} y^3\,dx + x^3\,dy$ where **C** is the circle of radius 2 centered at the origin.

With $P(x,y) = y^3$ and $Q(x,y) = x^3$ we have:

$$\begin{aligned}\oint_{\mathbf{C}} y^3\,dx + x^3\,dy &= 3\int\int_{x^2+y^2\leq 2} x^2 + y^2\,dx\,dy\\ &= 3\int_{\theta=0}^{2\pi}\int_{r=0}^{r=2} r^2\,r\,drd\theta\\ &= \frac{3}{4}\int_{\theta=0}^{2\pi} |_{r=0}^{2} r^4\,d\theta\\ &= 24\pi\end{aligned}$$

Where, because we are integrating over a circle we used polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ with the Jacobian being r.

<u>Point</u>: Once we have used the theorem we can still use change of coordinates to evaluate the resultant integral.

(c) $\int_{\mathbf{C}} y^3\,dx + x^3\,dy$ where **C** is two concentric circles: $\mathbf{C_2}$ and $\mathbf{C_1}$ centered at the origin – one of radius 2, the other of 1.

Remember our line integral must be over a closed path. Say we go completely around the inner circle $\mathbf{C_1}$, beginning and ending at a point p. How do we get to $\mathbf{C_2}$? Well, we just add any curve connecting p to $\mathbf{C_2}$, say at q. We travel along this curve to q and follow $\mathbf{C_2}$ around to q again. So we have now traversed $\mathbf{C_1}$ and $\mathbf{C_2}$, but at the cost of traveling along this extra connecting curve. But wait! We have to end up at p again, so we now travel along this curve from q to p, opposite the direction we first traveled it. So the net contribution of this path to the line integral is 0. With our closed line integral we can safely use Green's Theorem. In fact the integral is exactly the same as the above except we integrate over the same circle with the deleted inner circle. This corresponds to simply changing the lower bound of integration on r.

$$\begin{aligned}\oint_{\mathbf{C}} y^3\,dx + x^3\,dy &= 3\int\int_{1\leq x^2+y^2\leq 2} x^2 + y^2\,dx\,dy\\ &= 3\int_{\theta=0}^{2\pi}\int_{r=1}^{r=2} r^2\,r\,drd\theta\\ &= \frac{3}{4}\int_{\theta=0}^{2\pi} |_{r=1}^{2} r^4\,d\theta\\ &= \frac{45\pi}{2}\end{aligned}$$

<u>Point</u>: How to deal with "disconnected" boundaries

(d) $A(D) = \frac{1}{2}\int_c x\,dy - y\,dx$

That is, if $P(x,y) = \frac{-y}{2}$ and $Q(x,y) = \frac{x}{2}$ then:

$$\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} = 1$$

which yields the required formula. That is, by taking the line integral of a closed curve with the vector field $F(x,y) = (-\frac{y}{2}, \frac{x}{2})$ we can calculate the area enclosed by that curve!

Let us take our favourite example: the ellipse. So c is the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The parameterization is $x(t) = a\cos t$ and $y(t) = b\sin t$. So $\mathbf{T} = \frac{dc}{dt} = (-a\sin t, b\cos t)$. Then:

$$\begin{aligned}
\oint_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} &= \oint_{\mathbf{c}} \mathbf{F}\cdot\mathbf{T}ds \\
&= \frac{1}{2}\int_{t=0}^{2\pi} (-b\sin t, a\cos t)\cdot(-a\sin t, b\cos t)\,dt \\
&= \frac{ab}{2}\int_{t=0}^{2\pi} \sin^2 t + \cos^2 t\,dt \\
&= ab\pi
\end{aligned}$$

3.

Theorem 2. *Stokes Theorem*

Let D be a surface in $\mathbb{R}^3$:

1) D can be parametrized

2) The positively oriented curve $c = \partial D$ completely encloses D

3) The vector field F is C^1 over D

then:

$$\int_c F\cdot ds = \int_c F\cdot N\,dt = \int\int_D \nabla\times F\,dA$$

Evaluate with $F = (x+y, 2x-z, y)$ with $\mathbf{C}$ the boundary of the triangle with vertices $(2,0,0), (0,3,0), (0,0,6)$. (This is Example 8.20 in the text).

We need the equation of the plane that intersects $(2,0,0), (0,3,0), (0,0,6)$ to find the required parametrization for the surface of the triangle in 3-space. How do we find it?

The equation of the plane is just given. The traditional way of finding the equation of the plane is described in the text on page 29. Given that the points are just the intercepts, an easier way, is just to realize that all three points must satisfy $ax + by + cz = 1$. So $6c = 1$ and $3b = 1$ and $2a = 2$. With $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$ we have $z = 6 - 3x - 2y$. Our parametrization is therefore $x = u, y = v$ and $z = f(u,v) = 6 - 3u - 2v$.

Calculate $\nabla \times F$ as:

$$(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\hat{\mathbf{i}} + (\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x})\hat{\mathbf{j}} + (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})\hat{\mathbf{k}}$$

or

$$(1-(-1))\hat{\mathbf{i}} + (0-0)\hat{\mathbf{j}} + (2-1)\hat{\mathbf{k}} = (2,0,1)$$

So:

$$\oint_{\mathbf{C}} (x + y\,dx + 2x - z\,dy + y\,dz) = \int\int_S (2,0,1) \cdot d\mathbf{S}$$

where:

$$\begin{aligned} d\mathbf{S} &= \mathbf{N} du\,dv = \|\mathbf{T_u} \times \mathbf{T_v}\| du\,dv \\ &= \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix} du\,dv \\ &= \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{pmatrix} du\,dv \\ &= (3,2,1)\,du\,dv \end{aligned}$$

So finally:

$$\int\int_S (2,0,1) \cdot d\mathbf{S} = \int\int_S (2,0,1) \cdot (3,3,1)\,du\,dv = 7 \int\int_S du\,dv = 21$$

The text "cheats" in the last line since the double integral to be evaluated is equal to the area of a triangle which in this case is 3.

4.

Theorem 3. *Gauss's Theorem*

Let D be a volume in $\mathbb{R}^3$:

1) D is composed of elementary regions

2) The boundary surface $S = \partial D$ completely encloses D

3) The vector field F is C^1 over D

then:

$$\int\int_S F \cdot dS = \int\int_S F \cdot N\,dA = \int\int\int_D \nabla \cdot F\,dV$$

Let $F = (xy, -\frac{y^2}{2}, z)$ and our surface:

$$\begin{aligned} z &= 4 - 3x^2 - 3y^2, & 1 \le z \le 4 \\ x^2 + y^2 &= 1, & 0 < z \le 1 \\ z &= 0. \end{aligned}$$

Key to problem is to use cylindrical coordinates (where $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$). Using Gauss's Theorem the integral becomes:

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{4-3r^2} r\,dz\,dr\,d\theta$$

where:

$$\begin{aligned}0 \leq & z \leq 4-3r^2\\ 0 \leq & r \leq 1\\ 0 \leq & \theta \leq 2\pi\end{aligned}$$

and

$$\nabla\cdot\mathbf{F} = y - y + 1 = 1$$

with:

$$\begin{aligned}dx\,dy\,dz &= \frac{\partial(x,y,z)}{\partial(r,\theta,z)}\,dr\,d\theta\,dz\\ &= \det\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z}\end{pmatrix}\\ &= \det\begin{pmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}dr\,d\theta\,dz\\ &= r\,dr\,d\theta\,dz\end{aligned}$$

Putting everything together, we have:

$$\begin{aligned}\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{4-3r^2} r\,dz\,dr\,d\theta &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r(4-3r^2)\,dz\,dr\,d\theta\\ &= \int_{\theta}^{2\pi} |_{r=0}^{1} 2r^2 - \frac{3r^4}{4}\,d\theta\\ &= \int_{\theta=0}^{2\pi} d\theta\\ &= \frac{5\pi}{2}\end{aligned}$$