Math 280 Tutorial 2: Derivatives

September 28, 2004

The partial derivative, $\partial F/\partial x_i(x_1, \ldots, x_m)$ is the rate of change of F at $(x_1, \ldots, x_m)$ in the direction of the positive x_i axis (at unit speed). Usually it suffices to regard the other variables as constant and differentiate normally, but sometimes the definition must be used:

$$\frac{\partial F}{\partial x_i}(x_1, \ldots, x_m) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_m) - f(x_1, \ldots, x_i, \ldots, x_m)}{h}$$

If F is vector valued, then we define the partial derivatives of each of the component functions.

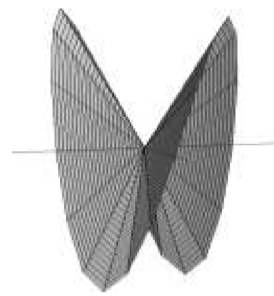
We can generalize this notion to find the rate of change at a point in any direction. For a 'nice' function, the rate of change will be a linear function of the direction vector. This means, among other things, that for $F : \mathbb{R}^m \to \mathbb{R}^n$, the rate of change at some point x_0 in any direction can be determined from knowing the rate of change at x_0 in m linearly independent directions. The m partial derivatives provide these m rates of change.

If a function is 'nice' at a point x_0 we call it differentiable at x_0. More precisely we say F is differentiable at x_0 if the matrix $DF(x_0)$ of partial derivatives at x_0 satisfies the following equation:

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - DF(x_0)(x - x_0)\|}{\|x - x_0\|} = 0 \tag{1}$$

If Equation (1) holds, then we say $DF(x_0)$ is the derivative of F at x_0. We say F is differentiable on an open set $U \subset \mathbb{R}^m$ if it is differentiable at every point in U.

Checking that (1) holds can be cumbersome, but fortunately we don't always have to do it. If all the partial derivatives exist at x_0 AND are continuous at x_0 then F is differentiable at x_0. Note that existence alone does not guarantee differentiability. Consider the following example. To construct the graph of this function, we start with the unit circle in the xy plane. Now we put some sort of oscillating function on the circle such that it is 0 on either of the coordinate axes, but not equal to 0 at any other points on the circle. Now fill in the rest of the function by connecting each point on the circle to the origin by a line starting at $(0, 0, 0)$ and ending at the correct height on the circle. Now the partials at $(0, 0)$ are both 0, since we made sure the function was 0 uniformly on both axes. So the derivative should be the zero-matrix. But if we approach 0 in any other direction, the rate of change is non-zero or undefined. See the figure below for the sketch of such a function.



Tutorial Problems and Solutions

1. Find the partial derivatives of the following functions.

   - $F(x, y) = x^y + y \ln x$

     $\partial F/\partial x = yx^{y-1} + y/x$
     $\partial F/\partial y = (lnx)x^y + \ln x$

   - $F(x, y) = \sin(e^{xy})$

     $\partial F/\partial x = ye^{xy} \cos(e^{xy})$
     $\partial F/\partial y = xe^{xy} \cos(e^{xy})$

   - $F(x, y, z) = \ln(x + y + z^2)$

     $\partial F/\partial x = (x + y + z^2)^{-1}$
     $\partial F/\partial y = (x + y + z^2)^{-1}$
     $\partial F/\partial z = 2z(x + y + z^2)^{-1}$

   - $F(x, y) = (x^4 + y^4)^{1/2}$

     $\partial F/\partial x = 2x^3(x^2 + y^2)^{-1}$ For $(x, y) \neq (0, 0)$
     $\partial F/\partial y = 2y^3(x^2 + y^2)^{-1}$ For $(x, y) \neq (0, 0)$
     For $(x, y) = (0, 0)$, we have to use the definition since the functions above are not defined at $(0, 0)$. Since $F(x, y) = F(y, x)$ we only have to do the calculation for one of the partials.
     $\partial f/\partial x(0, 0) = \lim_{h\to 0} \frac{F(h,0)-F(0,0)}{h} = \lim_{h\to 0} h^2/h = \lim_{h\to 0} h = 0$
     So both partials are 0 at $(0, 0)$.

2. Let $F(x,y,z) = \dfrac{xyz}{x^2+y^2+z^2}$ when $(x,y,z) \neq (0,0,0)$ and $F(0,0,0) = 0$. Where is F differentiable?

   F is differentiable at a point if all of its partials exist and are continuous at that point. Note that $F(x,y,z) = F(y,x,z) = F(x,z,y)$, so we only need to compute one partial.

   $$\frac{\partial F}{\partial x} = \frac{yz(x^2+y^2+z^2) + 2x^2yz}{(x^2+y^2+z^2)^2}$$

   This is continuous everywhere but the origin, and so the y and z partials will also be continuous everywhere but the origin. To see if F is differentiable at the origin we must use the definition of differentiability. First we compute the matrix of partials.

   $$\frac{\partial F}{\partial x}(0,0,0) = \lim_{h\to 0} \frac{0/h^2 - 0}{h} = 0$$

   By symmetry, the y and z partials are 0 at the origin, so $DF(0,0,0) = [0\ 0\ 0]$. Now we must check to see if (1) holds:

   $$\lim_{(x,y,z)\to(0,0,0)} \frac{\|F(x,y,z) - F(0,0,0) - DF(0,0,0)(x,y,z)\|}{\|(x,y,z)\|}$$
   $$= \lim_{(x,y,z)\to(0,0,0)} \frac{\left|\frac{xyz}{x^2+y^2+z^2}\right|}{(x^2+y^2+z^2)^{1/2}}$$
   $$= \lim_{(x,y,z)\to(0,0,0)} \frac{|xyz|}{(x^2+y^2+z^2)^{3/2}}$$

   If we approach along the line xyz we get $\lim_{x\to 0} |x^3|/(3x^3)$ which does not exist since it is $-1/3$ if we approach from the left and $1/3$ if we approach from the right. So F is differentiable on $\mathbb{R}^3/\{0,0,0\}$.

3. What is the rate of change at $(1,1,0)$ of the function $F(x,y,z) = (\ln(x^2 + y^2+z^2), 2xy+z)$ in the direction $(2,1,-1)$?

   First we must compute the derivative $DF(1,1,0)$:

   $$\begin{bmatrix} \frac{2x}{x^2+y^2+z^2} & \frac{2y}{x^2+y^2+z^2} & \frac{2z}{x^2+y^2+z^2} \\ 2y & 2x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

   Now we multiply by the direction vector:

   $$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

4. For a function $F : \mathbb{R}^3 \to \mathbb{R}$, we have that the rate of change at $(0,0,0)$ in the direction $(2,3,0)$ is -1, in the direction $(1,1,1)$ is 1, in the direction $(-1,1,1)$ is -1 and in the direction $(2,1,1)$ is 3. Show that F is not differentiable at $(0,0,0)$.

   Suppose F is differentiable at $(0,0,0)$. Then there exists a 1×3 matrix $DF(0,0,0)$ such that $DF(0,0,0)(2,3,0) = -1$, $DF(0,0,0)(1,1,1) = 1$, $DF(0,0,0)(-1,1,1) = -1$ and $DF(0,0,0)(2,1,1) = 3$. This can be written as:

$$DF(0,0,0)\begin{pmatrix} 2 & 1 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 & 3 \end{pmatrix}$$

   Taking the transpose of both sides gives the more familiar:

$$\begin{pmatrix} 2 & 3 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} DF(0,0,0)^T = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 3 \end{pmatrix}$$

   Row reducing the associated augmented matrix gives the following:

$$\left(\begin{array}{ccc|c} 2 & 3 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 2 & 1 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & -3 & -2 \end{array}\right)$$

   We see at this point that the last two equations cannot simultaneously hold and therefore no such $DF(0,0,0)$ can exist.

5. I have a differentiable function $F : \mathbb{R}^2 \to \mathbb{R}$ and I know that at $(0,0)$ the rate of change in the direction $(1,1)$ is 2 and in the direction $(-1,2)$ is 1. Find $DF(0,0)$.

   Following the same procedure as in the previous question, we get the following:

$$\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} DF(0,0)^T = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

   Multiplying by the inverse of the matrix on the left we get:

$$DF(0,0)^T = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   So $DF(0,0) = [1 \ \ 1]$.

6. Let $F(x,y)x^2 + y^3$. In what direction from the point $(1,1)$ would I have to travel at unit speed to get the rate of change to be $-1/\sqrt{2}$?

First we compute $DF(1,1) = [2\ 3]$. Now the rate of change in direction (a,b) is $DF(1,1)(a,b)^T = (2,3)\cdot(a,b) = |(2,3)||(a,b)|\cos\theta$ where θ is the angle between $(2,3)$ and (a,b). Now we know that $|(a,b)| = 1$. So we put $DF(1,1)(a,b)^T = -1/\sqrt{2}$ and we get $\sqrt{13}\cos\theta = -1/\sqrt{2}$. Now let $\alpha = \arctan(3/2)$, $\beta = \alpha + \theta$ and $\gamma = \alpha - \theta$. Then the two unit speed directions that give us a rate of change of $-1/\sqrt{2}$ are $(\cos\beta, \sin\beta) = (-0.9247\ldots, 0.3807\ldots)$ and $(\cos\gamma, \sin\gamma) = (1/\sqrt{2}, -1/\sqrt{2})$. This question could also be done solving the simultaneous system of equations $2x + 3y = -1/\sqrt{2}$, $x^2 + y^2 = 1$.