

One of the most powerful tools for differentiating functions is the chain rule. We are all familiar with the one dimensional version,  $\frac{d}{dx}(f \circ g)(x) = \frac{df}{dx}(g(x))\frac{dg}{dx}(x)$  and this generalizes to the multivariable case. Suppose we have a function  $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and another function  $\mathbf{G} : \mathbb{R}^m \rightarrow \mathbb{R}^l$ . Then the composition,  $\mathbf{G} \circ \mathbf{F}$  is defined. Now if  $\mathbf{F}$  is differentiable on an open subset  $U$  of  $\mathbb{R}^n$  containing a point  $\mathbf{a}$  and  $\mathbf{G}$  is differentiable on an open subset of  $\mathbb{R}^m$  containing  $\mathbf{F}(U)$  then  $\mathbf{G} \circ \mathbf{F}$  is differentiable at  $\mathbf{a}$ , and:

$$D(\mathbf{G} \circ \mathbf{F})(\mathbf{a}) = D\mathbf{G}(\mathbf{F}(\mathbf{a})) \cdot D\mathbf{F}(\mathbf{a})$$

Here  $\cdot$  denotes matrix multiplication. If we apply this to two single variable functions we get the traditional chain rule.

This rule can be useful in many situations, and can even be used to derive many other differentiation rules. For example, if you let  $\mathbf{F} : \mathbb{R} \rightarrow \mathbb{R}^2$  such that  $\mathbf{F}(x) = (f(x), g(x))$  and  $G : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $G(u, v) = uv$  then applying the chain rule to  $G \circ \mathbf{F}$  yields the well known product rule from single variable calculus.

Many times a function can be decomposed into the composition of functions. When computing the derivatives of such functions, we can either try to attack it as is, likely using the single variable chain rule when computing the partials, or we can decompose it and use the multivariable chain rule. Either approach will give the same results.

## Tutorial Problems

1. Let  $\mathbf{F}(x, y) = (f(g(x), h(y)), k(x, y))$ , where  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ . Assume all functions are differentiable. Find  $\mathbf{F}_x$  and  $\mathbf{F}_y$ .
2. Let  $F(x, y) = x^3y$ , where  $x^3 + tx = 8$  and  $ye^y = t$ . Find  $\frac{dF}{dt}(0)$ .
3. For  $\mathbf{F}, \mathbf{G} : \mathbb{R} \rightarrow \mathbb{R}^2$ , both differentiable, find  $D(\mathbf{F} \cdot \mathbf{G})$  where  $\cdot$  denotes the dot product in  $\mathbb{R}^2$ .
4. I have a differentiable function  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ . I switched into polar coordinates and found that at  $(r, \theta) = (1, \Pi/4)$  we have  $\partial F / \partial r = 2$  and  $\partial F / \partial \theta = -1$ . What are the partials with respect to  $x$  and  $y$  at the point  $(1/\sqrt{2}, 1/\sqrt{2})$ ?
5. I have a function that measures temperature and pressure at a point in  $\mathbb{R}^3$ , given by  $F(x, y, z) = (-z^{1/2} + \sin(x + y), 1/z^3 + e^{-xy})$ . I want to know what the rates of change are as I move along the surface  $z = e^{-xy}$ . Find the partial derivatives with respect to  $x$  and  $y$ .
6. Let  $f(x, y) = (e^{xy + \sin y}, e^{xy} \sin y)$ . Calculate  $Df$  first directly from the definition, then by writing  $f$  as the composition of two functions.