Math 110 — Homework Assignment 1

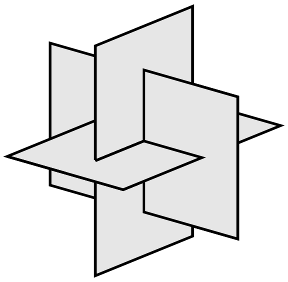
DUE DATE: SEPT. 20, 2005

1. The *icosahedron* is one of the prettiest solids discovered by the ancient Greeks. It has twelve vertices, thirty edges, and twenty faces made up of equilateral triangles. Imagining it without a drawing is a bit difficult, and part of Euclid's *Elements* is devoted to justifying that it even exists.

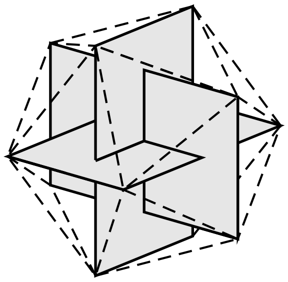
Let v be a vector connecting the center of the icosahedron to one of the vertices, and w be a vector connecting the center to any one of the five vertices near the one you picked. for v. Find:

(a) An expression for $\cos\theta$, where θ is the angle between v and w, and

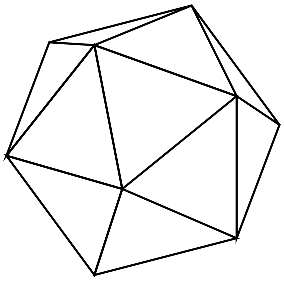
(b) The value of θ in degrees to at least three decimal places.



1



2



3

It might help to know the following construction of the icosahedron, due to Luca Pacioli, a friend of Leonardo da Vinci's, which first appeared in the 1509 book *De divina proportione*: Take three golden rectangles (i.e., rectangles whose proportions of side lengths are 1 to $(1+\sqrt{5})/2$) and interlock them to form the picture in the first figure. Then the twelve vertices of these rectangles form the twelve vertices of the icosahedron.

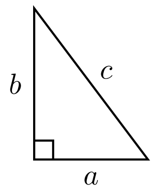
In part (a) of the question, give an algebraic expression for $\cos\theta$ (which will involve $\sqrt{5}$'s) rather than a real number with decimals.

2. Suppose we have a plane given in the form

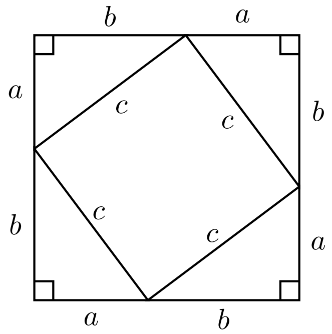
$$ax + by + cz = d.$$

in $\mathbb{R}^3$. Find and prove (i.e., give an argument for) a formula in terms of a, b, c, and d for the distance from the plane to the origin in $\mathbb{R}^3$. By distance from a plane to the origin, we mean the distance from the closest point on that plane to the origin.

3. To complete our proof of the formula for the dot product, I'd like to prove one of the last things we used: the Pythagorean theorem. So, starting with a right angled triangle with sides of length a, b, and c, (with c being the one opposite the right angle), like the one at right, we want to prove that $a^2 + b^2 = c^2$.



Here is the most elegant and famous proof of that fact. Arrange four copies of the triangle above so that they fit together to form a square of side length $a + b$. The middle of this big square will contain a smaller (twisted) square of side length c, as in the diagram below:



Now, calculate the area of the big square in two ways. First, by using the fact that its a square of side length $a + b$, and second, by adding the area of the square in the middle to the area of the four triangles. Compare the two answers to prove the Pythagorean theorem.