DUE DATE: Nov. 24, 2005

1. The following are ISBN codes with a single missing digit. Fill in the blanks to get the correct code. (Recall that we think of the code as a vector $\vec{v}$ in $\mathbb{F}_{11}^{10}$. A correct code is one for which $\vec{v} \cdot \vec{w} = 0$ in $\mathbb{F}_{11}$, where $\vec{w} = (10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$).

(a) 038_902791

(b) 01_8531710

(c) 354022_315

(d) 8_21302126

2. Computing in $\mathbb{F}_5$, find the general solution (i.e., possibly involving parameters) to the system of equations

$$\begin{array}{rcrcrcl} x & + & 4y & + & 3z & = & 3 \\ 3x & + & y & + & z & = & 2 \\ 2x & + & 2y & + & 3z & = & 4. \end{array}$$

3. To show that working with finite fields really is a lot like working with the field $\mathbb{R}$ that we're used to:

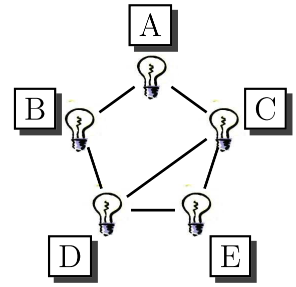
(a) Computing in $\mathbb{F}_{13}$, find a number y in with $y^2 = 3$ (for instance, by trial and error). This shows that it's possible to find a number which plays the role of $\sqrt{3}$ in $\mathbb{F}_{13}$.

(b) Use the quadratic formula to find the roots of $x^2 + 10x + 8$ in $\mathbb{F}_{13}$. Check your answer by plugging the values back into the polynomial.

(c) Find the roots of $7x^2 + 3x + 13$ in $\mathbb{F}_{17}$.

(d) Does $x^2 + 3x + 4$ have a root in $\mathbb{F}_5$? What goes wrong?

4. Consider the diagram of light-bulbs shown at right; The light-bulbs are labelled A through E, and a bulb can either be on  or off . There is also a switch for each letter. Pressing a switch changes the status of all the bulbs connected to the bulb with that letter (but excluding the bulb with that letter). For example, pressing switch A will switch the status of bulbs B and C, but leave A, D, and E alone. Pressing switch C will change the status of bulbs A, D, and E, but leave B and C alone. At the beginning, all bulbs are off.



(a) If we press switches A and D, what state are the bulbs left in?

(b) Represent the state of the light-bulbs by a vector $(x_a, x_b, x_c, x_d, x_e)$ in $\mathbb{F}_2^5$, with a 0 in an entry representing that the light-bulb is off, and a 1 representing that it is on. What is the vector representing the starting state, with all bulbs off? What is the vector representing the state from part (a)?

(c) The act of pressing switch A can be represented by adding a vector $\vec{v}_a$ to the vector representing the state of the system. What is $\vec{v}_a$?

(d) Similarly, work out the vectors $\vec{v}_b$, $\vec{v}_c$, $\vec{v}_d$, and $\vec{v}_e$ which represent the action of pushing switches B, C, D, and E.

(e) What is the dimension of the subspace spanned by $\vec{v}_a$, $\vec{v}_b$, $\vec{v}_c$, $\vec{v}_d$, and $\vec{v}_e$ in $\mathbb{F}_2^5$? Explain your answer carefully.

(f) Starting with the light-bulbs all off, can we get to any combination of the light-bulbs being on or off by pressing the switches? Why or why not?

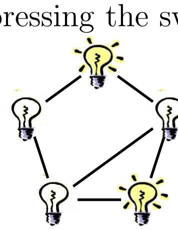
(g) Find coefficients u_a, …, u_e in $\mathbb{F}_2$ so that the solutions to

$$u_a x_a + u_b x_b + u_c x_c + u_d x_d + u_e x_e = 0$$

are exactly the subspace spanned by $\vec{v}_a$, $\vec{v}_b$, $\vec{v}_c$, $\vec{v}_d$, and $\vec{v}_e$.

(h) Given two states $(x_a, x_b, x_c, x_d, x_e)$ and $(y_a, y_b, y_c, y_d, y_e)$ there is a single algebraic condition (involving the x's and y's) that tells us whether or not we can get from the first state to the second state by pressing the switches. What is that condition? Explain your reasoning.

(i) Starting with the first state below, which of the other three can we get to by pressing the switches? Explain by using the answer from part (h).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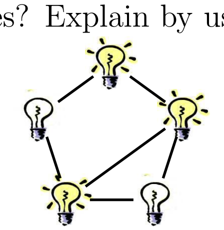
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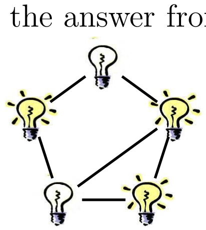
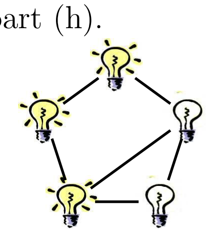

III


IV