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Here are some counting problems. Parts (a) and (b) are warm-up problems and don't have anything to do with the other three, but (c), (d), and (e) are vaguely related.

All the counting problems are trying about counting matrices with entries in $\mathbb{F}_p$, so all the answers will be formulas involving p.

(a) Using numbers in $\mathbb{F}_p$, how many matrices are there of the form $\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}$?

(b) How many 2×4 matrices are there (with entries in $\mathbb{F}_p$) of the form

$$A = \left[\vec{v}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \vec{w}_1 \,\middle|\, \vec{w}_2 \right]$$

(i.e., $\vec{v}_1$, $\vec{v}_2$, $\vec{w}_1$, and $\vec{w}_2$ are the column vectors) where $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, but $\vec{w}_1$ and $\vec{w}_2$ can be anything at all?

(c) Suppose that $\vec{v}_1$ is a nonzero vector in $\mathbb{F}_p^n$. How many vectors are there on the (1-dimensional) subspace spanned by $\vec{v}_1$?

(d) Suppose that $\vec{v}_1$ and $\vec{v}_2$ are two linearly independent vectors in $\mathbb{F}_p^n$. How many vectors in $\mathbb{F}_p^n$ are there which are either on the subspace spanned by $\vec{v}_1$ or the subspace spanned by $\vec{v}_2$ (or both)? [NOTE: this is not the same as the number of vectors on the plane spanned by $\vec{v}_1$ and $\vec{v}_2$ – we just want to count vectors that are on one line or the other].

(e) When making an error-correcting code, we start with a matrix M which has the property that for any pair of column vectors, the two vectors are linearly independent. (Then to create the code, we look at the kernel of M and find a basis $\vec{v}_1$, $\dots$, $\vec{v}_r$ and use them to transmit our message). Forgetting about the rest of that, I'd just like to try and count how many matrices there are with that property.

How many 3×3 matrices are there with that property? i.e., how many 3×3 matrices are there of the form

$$M = \left[\vec{v}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \vec{v}_3 \right]$$

where $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, $\vec{v}_2$ and $\vec{v}_3$ are linearly independent, and $\vec{v}_1$ and $\vec{v}_3$ are linearly independent. (Which doesn't necessarily mean that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ all together are linearly independent). All the entries are in $\mathbb{F}_p$, as usual.

BONUS QUESTION: How many $3 \times m$ matrices are there with this property? And what's the largest m for which this number is not zero? (i.e., for which there is a $3 \times m$ matrix with this property?).