DUE DATE: OCT. 27, 2005

1. Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an invertible linear transformation (i.e., it's bijective and all that). This means that it's possible to define the inverse function T^{-1}, a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, by

$$T^{-1}(\vec{w}) = \text{the unique vector } \vec{v} \text{ in } \mathbb{R}^n \text{ with } T(\vec{v}) = \vec{w}.$$

That certainly defines a *function* from $\mathbb{R}^n$ to $\mathbb{R}^n$, but we never checked that this function would also be a linear transformation. So: Prove that T^{-1} is a linear transformation, using the fact that T is. (As a perhaps puzzling hint, which you're free to ignore, if you're ever trying to prove that two vectors $\vec{v}_1$ and $\vec{v}_2$ are equal, it's enough to apply T to both sides and check that $T(\vec{v}_1) = T(\vec{v}_2)$, since T is injective.)

2. Now that we know that T^{-1} is a linear function, we know that it can be described by a matrix. In class we saw an algorithm to compute the inverse. The purpose of this question is to figure out why that algorithm works. The question is about a specific example in the 2×2 case, but the idea works for the general $n \times n$ case too.

(a) Find the RREF of the matrix $\left[\begin{array}{cc|ccc} 3 & 7 & 4 & 2 & 3 \\ 2 & 5 & 1 & 3 & 5 \end{array}\right]$.

(You can ignore the dots, I just felt like putting them in.)

(b) If A is the matrix $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$, find solutions to the linear equations $A\vec{x} = \vec{b}$ for the vectors $\vec{b}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\vec{b}_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ (i.e., you should find solutions to these three different systems of equations).

(c) If B is the inverse matrix for A, with column vectors $\vec{v}_1$ and $\vec{v}_2$, (so $B = \left[\vec{v}_1 \,\vdots\, \vec{v}_2\right]$), explain why $A\vec{v}_1 = \vec{e}_1$ and $A\vec{v}_2 = \vec{e}_2$. The kind of thinkng I had in mind was this: B is supposed to be the matrix for the transformation that's inverse to the transformation for A, and maybe thinking about what the inverse is supposed to do, and how you figure out the matrix for a linear transformation would lead to an explanation.

(d) Explain why putting the matrix $\left[\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]$ into RREF gives you $\left[I_2 \,\vdots\, B\right]$.

That is, why does the second matrix have to be B, and not any other matrix? I guess I'm really asking: why does this method of computing the inverse work?

3. Suppose that V and W are subspaces of $\mathbb{R}^n$. Show that the intersection of V and W (symbol: $V \cap W$) is also a subspace of $\mathbb{R}^n$. The intersection means all the vectors of $\mathbb{R}^n$ which are in both V and W.

4. Suppose that A is the matrix

$$A = \begin{bmatrix} -17 & -1 & 12 & 4 \\ -2 & 0 & 1 & 1 \\ -30 & -2 & 21 & 7 \\ 19 & 2 & -13 & -4 \end{bmatrix}$$

(a) Find a basis for $\ker(A)$. What is the dimension of $\ker(A)$?

(b) Find a basis for $\mathrm{im}(A)$. What is the dimension of $\mathrm{im}(A)$?

(c) What is the dimension of $\mathrm{im}(A) \cap \ker(A)$? (Provide details of your argument).

(d) What is the dimension of $\ker(A^2)$?

(e) Do the answers to (a), (b), and (c) imply the answer to (d)?

5. *Kernel puzzlers*

(a) Consider a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. If $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent vectors in $\mathbb{R}^n$, are the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_k)$ necessarily linearly dependent in $\mathbb{R}^m$? If so, why?

(b) If A is an $n \times p$ matrix, and B is a $p \times m$ matrix, with $\ker(A) = \mathrm{im}(B)$, what can you say about the product AB?

(c) if A is a $p \times m$ matrix, and B a $q \times m$ matrix, and we make a $(p+q) \times m$ matrix C by "stacking" A on top of B:

$$C = \begin{bmatrix} A \\ B \end{bmatrix},$$

what is the relation between $\ker(A)$, $\ker(B)$, and $\ker(C)$?

(d) Consider a square matrix A with $\ker(A^2) = \ker(A^3)$. Is $\ker(A^3) = \ker(A^4)$? Justify your answer. (Note, A^2 is shorthand for the matrix product AA, A^3 for AAA, etc.)

Note: This assignment is once again due on **Thursday**.