1. Suppose that $T_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ are linear transformations.

(a) If T_1 and T_2 are injective, prove that $T_2 \circ T_1$ is injective.

(b) If T_1 and T_2 are surjective, prove that $T_2 \circ T_1$ is surjective.

(c) If T_1 and T_2 are invertible, prove that $T_2 \circ T_1$ is invertible too.

2. From the definition, it's not clear that dimension is really a well defined notion. I guess I mean that if V is a subspace of $\mathbb{R}^n$ how do I know that every basis for V has the same number of vectors in it? Why couldn't one basis have 3 vectors and another basis have 5 vectors? If we're going to define the dimension of a subspace V as the number of vectors in any basis for V, we'd better know that the number doesn't depend on the basis.

So:

(a) If V is a subspace of $\mathbb{R}^n$, and v_1, v_2,..., v_q vectors in V which are linearly independent, and w_1, w_2, ..., w_p vectors in V which span V, show that $q \leq p$.

HINT: Let A be the $n \times q$ matrix whose columns are v_1,..., v_q, and B the $n \times p$ matrix whose columns are the vectors w_1,..., w_p. Since the w's span V, and since the v's are in V, explain why this means that there must be a matrix M with $A = BM$.

If $q > p$, explain why there has to be a nonzero vector in the kernel of M, and why this means that there has to be a nonzero vector in the kernel of BM. Explain why this would contradict the fact that the v's are linearly independent, and that therefore we must have $q \leq p$.

(b) If v_1, ..., v_q are a basis of V, and if w_1, ..., w_p are another basis of V, show that $p = q$, i.e., show that any two bases of V have the same number of vectors, and hence that the number $\dim(V)$ is well defined.

3. Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation.

(a) If $\vec{x}$ is a vector in $\mathbb{R}^n$, and $\vec{v}$ a vector in the kernel of T, explain why $T(\vec{x}) = T(\vec{x}+\vec{v})$.

(b) Conversely, if $\vec{x}_1$ and $\vec{x}_2$ are vectors in $\mathbb{R}^n$, and if $T(\vec{x}_1) = T(\vec{x}_2)$, explain why there is a vector $\vec{v}$ in the kernel of T with $\vec{x}_2 = \vec{x}_1 + \vec{v}$. (HINT: What does T do to $\vec{x}_2 - \vec{x}_1$?)

(c) If $\vec{b}$ is a vector in $\mathbb{R}^m$, and $\vec{x}_1$ a vector in $\mathbb{R}^n$ with $T(\vec{x}_1) = \vec{b}$, explain why the solutions to the equation $T(\vec{x}) = \vec{b}$ are all of the form $\vec{x} = \vec{x}_1 + \vec{v}$, with $\vec{v}$ in $\ker(T)$.

4. Suppose that T is the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by

$$T(x, y) = (6x - 3y,\, 2x - y,\, 4x - 2y).$$

Notice that the image of T is the line in $\mathbb{R}^3$ spanned by $(3, 1, 2)$.

(a) Find and describe the kernel of T.

(b) Find all the points in $\mathbb{R}^2$ that map to $(3, 1, 2)$ under T.

(c) Find all the points in $\mathbb{R}^2$ that map to $(-6, -2, -4)$ under T.

(d) Sketch (in $\mathbb{R}^2$), the answers from parts (a), (b), and (c).

(e) Explain how your sketch relates to the computations in question (3) above.

5. Suppose that A is a 3×2 matrix, and B is a 2×3 matrix.

(a) Explain why the 3×3 matrix AB can never be invertible. (HINT: what are the possibilities for the dimension of the kernel of AB?)

(b) Find an example where the 2×2 matrix BA *is* invertible.

(c) If BA is invertible, what must be the dimension of $\ker(A)$, and what must be the dimension of $\ker(B)$? Explain why.