1.

(a) If V is a subspace of $\mathbb{R}^n$, what does it mean that vectors $\vec{v}_1, \ldots, \vec{v}_k$ form a *basis* for V?

(b) Prove or disprove: the vectors $\vec{v}_1 = (2,3)$ and $\vec{v}_2 = (1,5)$ form a basis for $\mathbb{R}^2$.

(c) Prove or disprove: the vectors $\vec{w}_1 = (1,9)$ and $\vec{w}_2 = (2,3)$ form a basis for $\mathbb{R}^2$.

NOTE: To prove that they form a basis, you have to prove that they satisfy all the conditions to be a basis. To disprove that they are form a basis, you just have to show that they fail any one of the conditions.

(d) How many bases can a subspace have?

(e) Let V be the set of vectors (x,y,z,w) in $\mathbb{R}^4$ which are the solutions to the equations:

$$\begin{aligned} 11x - 9y + 69z + 59w &= 0 \\ -6x + 5y - 38z - 33w &= 0 \end{aligned}$$

The subset V is actually a subspace of $\mathbb{R}^4$. Find a basis for V.

2. The linear transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is given by the matrix

$$A = \begin{bmatrix} 2 & 0 & 6 & 1 \\ 3 & 1 & 11 & 0 \\ -3 & 0 & -9 & 1 \end{bmatrix}, \text{ which has RREF } \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is something that you don't have to prove.

(a) Find a basis for the image of T.

(b) The vector $\vec{b} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ is in the image. Find the linear combination of basis vectors from (a) which gives $\vec{b}$. Use this to find a vector $\vec{v}$ in $\mathbb{R}^4$ with $T(\vec{v}) = \vec{b}$.

(c) Find a basis for the kernel of T.

(d) Write down all the solutions to $T(\vec{v}) = \vec{b}$, with $\vec{b}$ the vector from part (b). You shouldn't have to do any complicated calculations to figure this out.

(e) Find all solutions to the system of equations

$$\begin{array}{rcrcrcrcl} 2x & + & & + & 6z & + & w & = & 5 \\ 3x & + & y & + & 11z & & & = & 5 \\ -3x & & & + & -9z & + & w & = & 0 \end{array}$$

using the "old" method – the way of solving equations that we learned in the first weeks of class.

(f) Explain the connection between (d) and (e).

3.

(a) Find all degree 3 polynomials $p(t) = a+b\,t+c\,t^2+d\,t^3$ with $p(1) = 3$ and $p(2) = -1$.

(b) In terms of a, b, c, and d, how do you express the condition that $p(1) = 0$?

(c) In terms of a, b, c, and d, how do you express the condition that $p(2) = 0$?

(d) Find a basis for the subspace V of $\mathbb{R}^4$ consisting of vectors $\vec{v} = (a, b, c, d)$ satisfying the equations in (b) and (c).

(e) Explain how the answer for (d) is connected to the answer for (a).