1.

(a) What does it mean for a subset V of $\mathbb{R}^n$ to be a subspace?

(b) If $a_1, \ldots, a_n$ are any numbers in $\mathbb{R}$, show that the solutions to the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ forms a subspace of $\mathbb{R}^n$.

(c) Suppose that we have more than one equation of the type from part (b), for instance, we might have k equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \qquad\qquad & \vdots \ \ \vdots \\
a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &= 0
\end{aligned}$$

where the a_{ij} are any numbers in $\mathbb{R}$. Show that the common solutions to those equations form a subspace of $\mathbb{R}^n$.

(d) The vectors $\vec{v}_1 = (-1, 3, 1, 2)$, $\vec{v}_2 = (2, 3, 2, -7)$, and $\vec{v}_3 = (2, 1, 1, -6)$ span a 3-dimensional subspace of $\mathbb{R}^4$. Find a single equation of the form $ax+by+cz+dw = 0$ whose solutions are this subspace.

2. The linear transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is given by the matrix

$$\begin{bmatrix} 2 & 1 & 2 & -1 \\ -1 & 2 & -1 & -2 \\ 1 & 4 & 1 & -4 \end{bmatrix}.$$

(a) Find a basis for $\ker(T)$ and a basis for $\operatorname{im}(T)$.

(b) The image in part (a) is 2-dimensional, which means that it is a plane in $\mathbb{R}^3$. Find an equation for this plane of the form $ax + by + cz = 0$.

(c) The kernel in part (a) is 2-dimensional, which means that it is a plane in $\mathbb{R}^4$. Find two equations of the form $ax + by + cz + dw = 0$ whose common solutions are this plane.

3.

(a) For each number 2, 3, …, 12 in $\mathbb{F}_{13}$ (i.e., working mod 13) find its multiplicative inverse, i.e., a number which when you multiply it by the original number you get 1 mod 13 (for instance: $4 \cdot 10 = 1 \pmod{13}$, so 10 is the multiplicative inverse of 4, although that also means that 4 is the multiplicative inverse of 10).

(b) Do the same for the numbers 2, 3, …, 16 in $\mathbb{F}_{17}$ (i.e., working mod 17).

4. Solve the following equations for x by "dividing" (i.e., multiplying by the multiplicative inverse). Show the details of your computation and check that your answer is correct by plugging it back into the equation.

(a) $2x = 5$ (in $\mathbb{F}_{13}$). (b) $2x = 5$ (in $\mathbb{F}_{17}$). (c) $2x = 5$ (in $\mathbb{F}_7$).

(d) $6x = 4$ (in $\mathbb{F}_{13}$). (e) $6x = 4$ (in $\mathbb{F}_{17}$). (f) $6x = 4$ (in $\mathbb{F}_7$).

5. One of the following sets of linear equations has a unique solution, and one has no solution. Decide which is which, find the solution to the equation where you can, and explain why the other equation has no solution. Consider both equations to be equations in $\mathbb{F}_{13}$.

(a) $\begin{aligned} 2x + 7y &= 4 \\ 5x + 3y &= 5 \end{aligned}$ (b) $\begin{aligned} 2x + 7y &= 4 \\ 4x + y &= 5 \end{aligned}$