MATH 110 Tutorial 2

Definition. Suppose $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ are vectors in $\mathbb{R}^n$. Then we define the *dot product* of $\vec{u}, \vec{v}$ by $\vec{u} \cdot \vec{v} = u_1 v_1 + \ldots u_n v_n$.

Definition. We say that two vectors $\vec{u}, \vec{v}$ are *orthogonal* if $\vec{u} \cdot \vec{v} = 0$.

Theorem. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$, and c a real scalar. Then

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$,
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$,
- $(c\vec{u}) \cdot \vec{v} = c(\vec{v} \cdot \vec{u})$,
- $\vec{u} \cdot \vec{u} \geq 0$,
- $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$.

Definition. We define *the norm* of a vector $\vec{x}$ as $|x| := \sqrt{\vec{x} \cdot \vec{x}}$.

Formula. $|\vec{u}| \ |\vec{v}| \ \cos\theta = \vec{u} \cdot \vec{v}$.

Definition. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be real vectors in n-space. The *projection* of $\vec{v}$ onto $\vec{u}$ is given by the equation

$$\operatorname{proj}_u v := \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right) \vec{u}.$$

1. (a) Project $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ onto the x-axis (in $\mathbb{R}^2$).

   (b) Project $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ onto the y-axis (in $\mathbb{R}^3$).

   (c) Project $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ onto the line $y = x$ (in $\mathbb{R}^2$).

2. For what real numbers k are the folowing pairs orthogonal?

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} k+1 \\ k-1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} k \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix}$$

3. If $|\vec{u}| = 2, |\vec{v}| = \sqrt{3}, \vec{u} \cdot \vec{v} = 1$, what is $|\vec{u} + \vec{v}|$?

4. Suppose we know that the projection of $\vec{u}$ onto $\vec{v}$ is zero. What does this mean geometrically? What if instead we had that it was $\vec{v}$?

5. Prove that $|\vec{u} \cdot \vec{v}| \leq |\vec{u}|\ |\vec{v}|$.

6. Prove that $\vec{u} + \vec{v} \ \perp \ \vec{u} - \vec{v}$ if and only if $|\vec{u}| = |\vec{v}|$. [What is this 'if and only if' ? You have to give two proofs! You must show that each implies the other! Sometimes we abbreviate 'if and only if' with 'iff'.]

7. Show that $\vec{u} \cdot \vec{v} = \frac{1}{4}|\vec{u} + \vec{v}|^{\,2} - \frac{1}{4}|\vec{u} - \vec{v}|^{\,2}$.

8. *Challenge.* How many different ways can you combine quarters and dimes to make \$2.80? Generate a nice way of listing the solutions $(\#quarters, \#dimes)$. Consider the question of making \$1432.45 out of quarters and dimes. What lets you write down the solution set easily?