MATH 110 Tutorial 2 Solutions

1. (a) Project $v := (1, 1)$ onto the x-axis (in $\mathbb{R}^2$).
   (b) Project $v := (1, 0, 1)$ onto the y-axis (in $\mathbb{R}^3$).
   (c) Project $v := (-1, 2)$ onto the line $y = x$ (in $\mathbb{R}^2$).

Solution. The *projection* of $\vec{v}$ onto $\vec{u}$ is given by the equation $\text{proj}_u v := \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$.

(a) Let $u := (a, 0)$ with $a \neq 0$, a vector on the x-axis. If you like, choose a concrete vector! $u \cdot u = a^2, u \cdot v = a$, and hence $\frac{u\cdot v}{u\cdot u}u = (1, 0)$. If you visualize this geometrically, you can see that dropping the perpendicular from the point $(1, 1)$ to the x-axis hits the point $(1, 0)$.

(b) Let $u := (0, a, 0)$ with $a \neq 0$, a vector on the y-axis. Again, if you'd rather, choose a concrete vector. $u \cdot u = a^2, u \cdot v = 0$, so $\frac{u\cdot v}{u\cdot u}u = (0, 0, 0)$.

(c) Let $u := (a, a)$, with $a \neq 0$, a vector on the $y = x$ line. You're still free to choose a specific vector instead. Here we have $u \cdot u = 2a^2, u \cdot v = a$. Hence $\frac{u\cdot v}{u\cdot u}u = \left(\frac{1}{2}, \frac{1}{2}\right)$.

2. For what real numbers k are the folowing pairs orthogonal?

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} k+1 \\ k-1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} k \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix}$$

Solution. Two vectors are orthogonal if and only if their dot product is zero.

(a) $(2, 3) \cdot (k + 1, k - 1) = 2k + 2 + 3k - 3 = 5k - 1$, which is zero if and only if k is one fifth.

(b) $(0, 2) \cdot (k, 0) = 0$, which is zero for all real k.

(c) $(1, -1, 2) \cdot (k^2, k, -3) = k^2 - k - 6$ which is zero if and only if $k = -2, 3$.

3. If $||\vec{u}|| = 2, ||\vec{v}|| = \sqrt{3}, \vec{u} \cdot \vec{v} = 1$, what is $||\vec{u} + \vec{v}||$?

Solution. By definition, $||\vec{u} + \vec{v}|| = \sqrt{(u + v) \cdot (u + v)} = \sqrt{u \cdot u + v \cdot v + 2(u \cdot v)}$. Remembering that $u \cdot u = ||u||^2 = 4$, and $v \cdot v = ||v||^2 = 3$, we have altogether $||u + v|| = \sqrt{4 + 3 + 2(1)} = 3$.

4. Suppose we know that the projection of $\vec{u}$ onto $\vec{v}$ is zero. What does this mean geometrically? What if instead we had that it was $\vec{v}$?

Solution. If the projection of v onto u is zero, then $u \cdot v = 0$ or u is zero (by the formula). In either case we have v is perpendicular to u. Suppose we have $\operatorname{proj}_u v = v$, then $\frac{u \cdot v}{u \cdot u} u = v$, and this means that v is a scalar multiple of u; hence when v lies along the same line as u.

5. Prove that $|\vec{u} \cdot \vec{v}| \leq ||\vec{u}||\ ||\vec{v}||$.

Solution. We have the formula $u \cdot v = ||u||\ ||v||\ \cos\theta$, with θ the angle between u, v. By taking absolute values, and remembering that norms are never negative, we have $|u \cdot v| = ||u||\ ||v||\ |\cos\theta|$. The cosine of a number is always between -1 and 1, so in absolute value cosine is always between 0 and 1; hence $|u \cdot v| = ||u||\ ||v||\ |\cos\theta| \leq ||u||\ ||v||$ as required.

6. Prove that $\vec{u} + \vec{v} \quad \perp \quad \vec{u} - \vec{v}$ if and only if $||\vec{u}|| = ||\vec{v}||$. [What is this 'if and only if' ? You have to give two proofs! You must show that each implies the other! Sometimes we abbreviate 'if and only if' with 'iff'.]

Solution. We will give two proofs, although they are essentially the same proof given in opposite directions - this is not always how so-called "biconditional" proofs are done.
$\Leftarrow$: Suppose first that $u + v \quad \perp \quad u - v$. Using this we will prove that $||\vec{u}|| = ||\vec{v}||$. Two vectors are orthogonal implies their dot product is zero:

$$\begin{aligned}
0 &= (u+v)\cdot(u-v) \\
&= u \cdot u - u \cdot v + v \cdot u - v \cdot v \\
&= u \cdot u + (u \cdot v - u \cdot v) - v \cdot v \\
&= u \cdot u - v \cdot v \\
u \cdot u &= v \cdot v \\
\sqrt{u \cdot u} &= \sqrt{v \cdot v} \\
||u|| &= ||v||
\end{aligned}$$

Note here that we do not need to concern ourselves with a sign problem (extraneous solution) because we know that norms are never negative. This gives us the first half of the proofs. $\Rightarrow$: Now we will suppose that $||u|| = ||v||$ and use this to deduce that $u+v$ and $u-v$ are orthogonal.

$$\begin{aligned}
||u|| &= ||v|| \\
\sqrt{u \cdot u} &= \sqrt{v \cdot v} \\
u \cdot u &= v \cdot v \\
0 &= u \cdot u - v \cdot v \\
&= u \cdot u + (u \cdot v - v \cdot u) - v \cdot v \\
&= u \cdot u - u \cdot v + v \cdot u - v \cdot v \\
0 &= (u+v) \cdot (u-v).
\end{aligned}$$

The dot product of two vectors being zero implies that they are orthogonal, and hence we have that $u+v \quad \perp \quad u-v$. This completes the required proof.

7. Show that $\vec{u} \cdot \vec{v} = \frac{1}{4}||\vec{u}+\vec{v}||\ ^2 - \frac{1}{4}||\vec{u}-\vec{v}||\ ^2$.

Solution.

$$\begin{aligned}
\frac{1}{4}||\vec{u}+\vec{v}||\ ^2 - \frac{1}{4}||\vec{u}-\vec{v}||\ ^2 &= \frac{1}{4}(u+v) \cdot (u+v) - \frac{1}{4}(u-v) \cdot (u-v) \\
&= \frac{1}{4}(u \cdot u + 2u \cdot v + v \cdot v - u \cdot u + 2u \cdot v - v \cdot v) \\
&= u \cdot v
\end{aligned}$$

8. *Challenge.* How many different ways can you combine quarters and dimes to make \$2.80? Generate a nice way of listing the solutions $(\#quarters, \#dimes)$. Consider the question of making \$1432.45 out of quarters and dimes. What lets you write down the solution set easily?

Comment. I will not give the solution here, because it of itself is not particularly illuminating. The interesting question is the last part. The crisp way of writing down the solutions is using the equation 2 Quarters - 5 Dimes = 0. Once we have found one solution, we can find all others by adding multiples of this relation, which is called a *kernel* equation. You'll see more about this in the days to come...