MATH 110 Tutorial 3 Solutions

1. Reduce the following matrices into reduced row echelon form.

$$(a)\begin{bmatrix} 1 & 0 & 8 \\ 2 & 3 & 3 \\ 4 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \quad (b)\begin{bmatrix} -2 & 4 & 2 \\ 3 & -1 & 1 \end{bmatrix} \quad (c)\begin{bmatrix} 1 & 2 \\ 4 & 0 \\ -2 & 1 \\ -1 & 1 \end{bmatrix} \quad (d)\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Solution.

$$(a)\begin{bmatrix} 1 & 0 & 8 \\ 2 & 3 & 3 \\ 4 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (b)\begin{bmatrix} -2 & 4 & 2 \\ 3 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{3}{5} \\ 0 & 1 & \frac{4}{5} \end{bmatrix},$$

$$(c)\begin{bmatrix} 1 & 2 \\ 4 & 0 \\ -2 & 1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (d)\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

2. For each of the above matrices, now consider them as augmented matrices, and write explicitly the solution set for the system of equations they represent.

Solution.

(a) There are no solutions. The third line corresponds to the nonequation $0 = 1$.

(b) The system has the unique solution $x = \frac{3}{5}, y = \frac{4}{5}$.

(c) There are no solutions. The second line corresponds to the nonequation $0 = 1$.

(d) There is a single pivot, and hence the solution set consists of the points (x_1, x_2, x_3, x_4) satisfying $x_1 = 1 - x_2 - x_3 - x_4$.

3. What is the rank of each matrix above? Does the rank help in deciding how many solutions there are to the corresponding system of equations?

Solution.

The rank is the number of pivots (*i.e.* leading ones) in the reduced row echelon form. Therefore the ranks are $3, 2, 2, 1$ respectively.

4. Solve the system of equations:

$$\begin{aligned} 4x + 2y - z &= 2 \\ -x + 3y - 2z &= 0 \\ 2x - 4y + 5z &= -1 \end{aligned}$$

Solution.

$$\begin{bmatrix} 4 & 2 & -1 & 2 \\ -1 & 3 & -2 & 0 \\ 2 & -4 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 15/32 \\ 0 & 1 & 0 & -14/32 \\ 0 & 0 & 1 & -18/32 \end{bmatrix}$$

Hence the system of equations has the unique solution $x = \frac{15}{32}, y = \frac{-14}{32}, z = \frac{-18}{32}$.

5. Prove that if a matrix has more columns than rows it has a free variable.

Solution.

See Assignment.

6. *Challenge.* Let A be a 2×2 matrix, and consider the system of equations represented by $A|0$. Determine a simple test for deciding when there is only the solution $x = y = 0$ to the implied system of equations. What about a test for 3×3 ?

   Solution.

   $$\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} ac & bc & 0 \\ ca & da & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} a & b & 0 \\ 0 & ad - bc & 0 \end{array}\right]$$

   if we suppose that a is non-zero. If a is zero, then swap the rows, and assume that c is non-zero. If c is zero, then the matrix cannot have two pivots, and $ad = bc$ anyways. Thus, the system has a unique solution **iff** $ad - bc \neq 0$. (This form, $ad - bc$, has a special name, the *determinant*). There is a form like this for 3×3, but no one bothers to remember it. :p