MATH 110 Tutorial 4

Definition. A *linear combination* of $v_1, \ldots, v_m$ is a sum $c_1v_1 + \ldots + c_mv_m$ for scalars $c_1, \ldots, c_m$. The vectors $v_1, \ldots v_n$ are said to be *linearly independent* if the only time $c_1v_1 + \ldots + c_nv_n = 0$ is when all of the $c_j = 0$.

Definition. The set of all possible linear combinations of the collection of vectors $v_1, \ldots, v_n$ is called the *span* of $v_1, \ldots, v_n$, which we denote as $\text{span}\{v_1, \ldots v_n\}$

Definition. A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is *linear* if for every $\vec{v}, \vec{w} \in \mathbb{R}^m$ and scalar k, it satisfies:

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$,
2. $T(k\vec{v}) = kT(\vec{v})$.

Observation. A linear transformation T preserves zero; $T(0) = 0$.

<u>Practice Problems</u>.

1. Determine whether $\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2$. If so, find the weights a, b so that $\vec{w} = a\vec{v}_1 + b\vec{v}_2$.

   (a) $\vec{v}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

   (b) $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

2. Let T be the function which sends a cubic polynomial to its second derivative. Show that T is a linear transformation.

3. Suppose $T_1, T_2, T_3, T_4 : \mathbb{R}^2 \to \mathbb{R}^3$ yield the following images.

$$
\begin{array}{r|r}
(x,y) & T_1(x,y) \\
--- & ---- \\
[0,0] & [1,0,0] \\
[11,0] & [1,2,-1] \\
[0,-6] & [-1,1,4]
\end{array}
,\qquad
\begin{array}{r|r}
(x,y) & T_2(x,y) \\
--- & ---- \\
[1,0] & [1,0,0] \\
[0,1] & [1,1,1] \\
[2,2] & [3,2,2]
\end{array}
,
$$

$$
\begin{array}{r|r}
(x,y) & T_3(x,y) \\
--- & ---- \\
[1,2] & [3,5,9] \\
[-1,2] & [1,0,1] \\
[1,1] & [1,1,1]
\end{array}
,\qquad
\begin{array}{r|r}
(x,y) & T_4(x,y) \\
--- & ---- \\
[5,2] & [1,2,3] \\
[1,2] & [0,1,2] \\
[3,-1] & [1,1,1]
\end{array}
.
$$

Which of these four maps *cannot* be linear? Hint: using the two properties of a linear map, we can use linear combinations...

4. Let $f(x,y) := x^2 + y^2$, and $g(x,y) := xy$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(x,y) = \begin{bmatrix} f(x+2, y+3) - f(x,y) - f(2,3) \\ g(x-1, y+5) - g(x,y) - g(-1,5) \end{bmatrix}.$$

Determine whether T is a linear transformation or not.

5. *Challenge.* Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. The collection of points in $\mathbb{R}^m$ whose image under T is $\vec{0}$ is called the *kernel* of T, and is denoted $\ker(T)$. Give an example of a linear transformation whose kernel is the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$. In general, suppose you are given a kernel as a linear equation as above. Find a linear transformation with this kernel.