MATH 110 Tutorial 5

- A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ can be expressed by the map $T(\vec{x}) = A\vec{x}$ for some $n \times m$ matrix A.

- The composition of any number of linear maps is linear. The order in which we compose linear maps is very important, since different orders give us different composite maps! We *define* matrix multiplication so that it corresponds to the composition of the linear map of the factors.

- Matrix multiplication AB is defined only if A has the same number of columns as B has rows (why?). The resulting matrix AB will have the same number of rows as A and the same number of columns as B. This means that if we write the dimensions of A, B side-by-side, we can only multiply if the middle two numbers are the same. In this case, the dimensions of AB are what is left when we cross out the middle matching numbers.

- A function $f : A \to B$ is said to be *injective* or *one to one* if at most one element of A is sent to the same B. A little more formally, this means that if $f(a_1) = f(a_2)$, then $a_1 = a_2$. Also, we say that f is *surjective* or *onto* if f hits every element of B. Formally this means that for each $b \in B$, there exists at least one (maybe more than one!) $a \in A$ so that $f(a) = b$. A map which is both injective and surjective is called *bijective*.

- In $\mathbb{R}^2$, rotations and reflections are examples of linear transformations from the space to itself. The matrix representing a **counter-clockwise** rotation of θ is given by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Practice Problems.

1. Find the matrix for the reflection in the line $y = x$ in $\mathbb{R}^2$. Also find the matrix for the rotation of $\pi/2$ in the counter-clockwise direction. Compute the matrix which is the composition of the rotation after the reflection by following the image of the standard basis vectors.

2. Compose a reflection in the line $y = x$ with a reflection in the y-axis. Again compute the matrix for the composition by following the image of the standard basis. Can you explain what you get?

3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be linear. Think about what linear maps from $\mathbb{R}$ to itself look like, and use this to prove that $f \circ g$ is linear.

4. Give an example of a nonzero linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ for which $f \circ f$ is the zero map. Write the matrix A for f in terms of the standard basis, and compute A. By computing $f \circ f(e_1), f \circ f(e_2)$, write the matrix for $f \circ f$. Does your answer make sense?

5. *Challenge*. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be linear. We call the set of $a \in A$ for which $f(a) = 0$ the *kernel* of f. Prove that f is injective if and only if the kernel of f is only zero. (HINT: Use the linearity of f !)