MATH 110 Tutorial 5 Solutions

1. For the reflection in $y = x$, the standard basis vectors $(1, 0) \mapsto (0, 1)$ and $(0, 1) \mapsto (1, 0)$. This gives us the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The matrix for a rotation is given by sticking in the angle to the general form given on the front page of the tutorial hand out; alternatively, the rotation sends $(1, 0) \mapsto (0, 1)$ and $(0, 1) \mapsto (-1, 0)$, so we have the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

For the composition, $(1, 0) \mapsto (0, 1) \mapsto (-1, 0)$ by reflecting and then rotating; meanwhile, $(0, 1) \mapsto (1, 0) \mapsto (0, 1)$. This gives us the matrix for the composition

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Of course it matters which you do first.. try both! What is the difference? Suppose we do the reflection in $y = x$ first. Then $(1, 0) \mapsto (0, 1) \mapsto (0, -1)$ and at the same time $(0, 1) \mapsto (1, 0) \mapsto (1, 0)$, giving us the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

If we had done them the other way around, with the reflection in $x = 0$ first, we'd have had $(1, 0) \mapsto (1, 0) \mapsto (0, 1)$ and at the same time $(0, 1) \mapsto (0, -1) \mapsto (-1, 0)$. This gives us the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Alright, they aren't the same thing – what's the big deal? These are the matrices of rotations! Yup, it's actually possible to duplicate a rotation by composing two reflections. These two are rotations in counterclockwise angles of $3\pi/2$ and $\pi/2$, respectively.

3. Linear maps from $\mathbb{R}$ to itself are maps $T(x) = cx$ for some real c. Why? they're given by 1×1 matrices. We can call $f(x) = cx$ and $g(x) = dx$ for real numbers c, d. Great, now $f \circ g(x) = f(g(x)) = f(dx) = df(x) = dcx = (dc)x$, and since dc is real, we have another linear map.

4. The map $T : x \mapsto 0$, $T : y \mapsto x$ works. Why does it work? Well, $T \circ T(x) = T(T(x)) = T(0) = 0$, and $T \circ T(y) = T(T(y)) = T(x) = 0$. What's the matrix for this map?

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

By what we got above, we know that the matrix for $T \circ T$ is given by

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

the zero matrix. It makes sense that we got the zero matrix which represents the zero transformation (send everything to zero). You can also double check that if I square the matrix given for my T that the product is zero.

5. Remember I owe you two proofs! First suppose that $T : \mathbb{R}^m \to \mathbb{R}^n$ is injective. Then it is immediate that the only thing that can be sent to zero is zero; if anything else is sent to zero, T wouldn't be injective. Alright, that wasn't very interesting. For the converse, suppose the kernel of T is only zero. Now suppose that $T(a) = T(b)$ for some $a, b \in \mathbb{R}^m$. We have to show that $a = b$, to verify injectivity. Let's use the linearity of T. We know that $T(a) = T(b)$ means that $T(a) - T(b) = \vec{(0)}$, or $T(a - b) = 0$. However, we said that the kernel of T was only zero, so the only vector that T sends to zero is zero. Hence $a - b = 0$ or $a = b$, as required.