MATH 110 Solutions 6

Solutions to Practice Problems.

$(a) \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{6} \end{bmatrix}$

$(b) \begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 2 \\ -1 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ -\frac{11}{3} & \frac{4}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

$(c) \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -1 \\ 2 & -4 & k \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 5-k \end{bmatrix}$

A matrix is invertible if and only if it is square and rrefs to the identity matrix, so this matrix is invertible for all real $k \neq 5$.

2. $\ker(A) = \ker(B) = \{0\}$ means that the only element that A or B sends to zero is zero. Let's suppose that x is in the kernel of AB. That means $(AB)x = 0$, or that AB sends x to zero. We can think of $(AB)x$ as the composite A after $B(x)$,

$$(AB)x = A(B(x)) = 0.$$

Here A is sending a vector to zero, and so that vector $B(x)$ must be zero because we know the kernel of A is only zero. Now we have $B(x) = 0$. Now B is sending x to zero, and so that vector must be zero since we know the kernel of B is only zero. Hence, we know that if x is in the kernel of AB, it is zero. Therefore the kernel of AB is zero.

3. Pick an invertible matrix A. Then $-A$ (the matrix whose entries are the negatives of the entries of A) is invertible – its inverse is $-(A^{-1})$. The sum $A + (-A) = 0$ is not invertible.

4. We can expand $A^3 = AAA$. The inverse of A^3 is $(A^{-1})^3$, since

$$A^3(A^3)^{-1} = AAAA^{-1}A^{-1}A^{-1} = AAA^{-1}A^{-1} = AA^{-1} = I_n.$$

Let B be nilpotent, so that $B^n = 0$ for some $n > 0$. Suppose that B is invertible. The idea from the first part of the question is that when a matrix is invertible, powers of that matrix are invertible. B^n has an inverse, namely $(B^{-1})^n$. Let's multiply both sides of $B^n = 0$ by this inverse, to get $(B^{-1})^n B^n = (B^{-1})^n 0$. The left-hand side is the identity matrix – it has to be, a matrix times its inverse is always the identity matrix. The right-hand side is zero – it has to be, any matrix times a zero matrix is zero. Hence, we have proven that the identity matrix is the zero matrix – a contradiction. Therefore a nilpotent matrix is never invertible.

5. Let w_1, w_2 be in the image of T. Then there exist two vectors v_1, v_2 so that $T(v_1) = w_1$ and $T(v_2) = w_2$ by the definition of image. In order to show that $w_1 + w_2$ is in the image, we need to find a vector that maps to it. It is immediate from the linearity of T that the vector $v_1 + v_2$ is such a vector:

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2.$$

For the second part, suppose that v_1, v_2 are in the kernel of T. Then from the definition of the kernel of a map, we know $T(v_1) = 0 = T(v_2)$. In order to check that $v_1 + v_2$ is in the kernel of T, we need to show that $T(v_1 + v_2) = 0$. This is immediate from the linearity of T:

$$T(v_1 + v_2) = T(v_1) + T(v_2) = 0.$$