MATH 110 Tutorial 7

A basis is a linearly independent spanning set. This means: If $\beta_1, \ldots \beta_n$ is a basis for $\mathbb{R}^n$ then the vectors $\beta_1, \ldots, \beta_n$ are linearly independent, and every $x \in \mathbb{R}^n$ can be written as $x = c_1\beta_1 + \ldots c_n\beta_n$ for some real scalars $c_1, \ldots c_n$.

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation represented by the matrix A. A basis for the image of T (or A) is the set of pivot columns of A. **Note:** It does *NOT* work to take the pivot columns from $\mathrm{rref}(A)$.

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation represented by the matrix A. A basis for the kernel of T (or A) is obtained by solving the system $Ax = 0$ in terms of the free variables.

A subspace V of $\mathbb{R}^n$ is a subset which:

1. Contains the zero vector: $\vec{0} \in V$,
2. is closed under scalar multiplication: For $c \in \mathbb{R}$ and $\vec{v} \in V$, we have $c\vec{v} \in V$,
3. is closed under vector addition: For $\vec{v_1}, \vec{v_2} \in V$, we have $\vec{v_1} + \vec{v_2} \in V$.

Practice Problems.

1. Find a basis for the kernel and image of the following matrices.

   (a) $\begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 1 \\ 3 & 3 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 4 \\ -2 & -1 \\ 3 & 4 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 1 & -2 & 6 \\ 0 & -2 & 4 & 2 \end{bmatrix}$.

2. Let A be an $n \times n$ matrix. Compare the kernel of A with the kernel of A^2.
3. Is the union of two subspaces a subspace?
4. Say something about the image of AB compared to the image of A.
5. Prove that every line through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.
6. *Challenge.* Suppose A is an $n \times n$ nilpotent matrix, i.e., some positive power of A is zero. Show that the rank of A is at most $n/2$.

Things you should know...

- Dot products
  - definition, computation
  - Orthogonality
  - Norms
  - Projection
  - $a \cdot b = \|a\| \, \|b\| \cos\theta$
- RREF
  - definition, computation
  - zero, one, infinitely many solutions to a system of linear equations
  - rank
- Linear Transformations
  - definition
  - injectivity, surjectivity
  - image, kernel
  - composition
  - matrix multiplication
  - inverting a matrix / linear transformation
- Linear combinations
  - linear independence
  - spanning set
  - bases
  - solving equations
- Subspaces
  - definition
  - identifying subspaces, *e.g.* kernel and image