Tutorial 7 Solutions

1. Find a basis for the kernel and image of the following matrices.

(a) $\begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 1 \\ 3 & 3 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 4 \\ -2 & -1 \\ 3 & 4 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 1 & -2 & 6 \\ 0 & -2 & 4 & 2 \end{bmatrix}$.

Solution. The rrefs are

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

In all cases, a basis for the image is the first two columns of each **original** matrix. A basis for the kernels is: (a) $(1, -1, 0)$, (b) $\vec{0}$, (c) $(-5, -1, 0, 1), (0, 2, 1, 0)$.

2. Let A be an $n \times n$ matrix. Compare the kernel of A with the kernel of A^2.

Solution. The kernel of A^2 is contained in the kernel of A; this should be clear, because if A sends something to zero, then AA sends it to zero and then zero to zero.

3. Is the union of two subspaces a subspace?

Solution. Not in general; consider two lines, and a vector on each. Their sum is not in the union.

4. Say something about the image of AB compared to the image of A.

Solution. The image of AB is contained in the image of A. The reason for this is that B may not hit 'enough'; A is defined on a whole space, whereas in AB, A only acts on those points hit by B.

5. Prove that every line through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

   Solution. The origin is in every line through the origin. The sum of two vectors on a line is on the line. Any scaling of a vector is along the same ray.

6. *Challenge.* Suppose A is an $n \times n$ nilpotent matrix, i.e., some positive power of A is zero. Show that the rank of A is at most $n - 1$.

   Solution. We saw in a previous tutorial that a matrix cannot be both nilpotent and invertible. A matrix is invertible if and only if it has full rank.