MATH 110 Tutorial 8

A basis is a linearly independent spanning set. A subspace of $\mathbb{R}^n$ has many bases.

The number of elements in a basis for a space is constant. This number is called the *dimension* of the space.

The *rank* of a transformation is the dimension of the image. Equivalently, it is the number of pivot columns. The *nullity* of a transformation is the dimension of the kernel. Equivalently, it is the number of non-pivot columns.

Theorem. (The Rank-Nullity Theorem): For a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, we always have

$$\text{rank}(T) + \text{nullity}(T) = n.$$

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. T is injective if and only if the kernel of T is $\vec{0}$.

Practice Problems.

1. Find three different bases for the subspace of $\mathbb{R}^3$ spanned by the solutions to the equation $x + y - z = 0$.

2. Suppose we have a linear transformation $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ surjects. What is the nullity of T_1 ? Generalize: If a linear map $T_2 : \mathbb{R}^m \to \mathbb{R}^n$ surjects, then what is the nullity of T_2?

3. Let A, x_1, x_2, y be as given below. Find a basis for the kernel of A. Next, note that $Ax_1 = y$ and $Ax_2 = y$. Check that $x_1 - x_2$ is in the kernel of A two different ways: first compute $A(x_1 - x_2)$, and second show that $x_1 - x_2$ is a linear combination of the basis you computed for the kernel of A.

$$A := \begin{bmatrix} 1 & 0 & 4 & 1 & -1 \\ 2 & 1 & 9 & 7 & -1 \\ 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 5 & 3 & -1 \end{bmatrix}, \quad x_1 := \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad x_2 := \begin{bmatrix} 5 \\ 5 \\ 1 \\ -1 \\ 5 \end{bmatrix}, \quad y := \begin{bmatrix} 3 \\ 12 \\ 2 \\ 7 \end{bmatrix}.$$