Tutorial 8 Solutions

1. Find three different bases for the subspace of $\mathbb{R}^3$ spanned by the solutions to the equation $x + y - z = 0$.

   Solution. We have $x = -y+z$ or the space is the span of the vectors $(-1, 1, 0), (1, 0, 1)$. Any two linear combinations of these two vectors that don't lie on the same line through the origin will work! I will also choose $(0, 1, 1), (3, 0, 3)$ and $(-2, 2, 0), (\pi, 0, \pi)$.

2. Suppose we have a linear transformation $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ surjects. What is the nullity of T_1 ? Generalize: If a linear map $T_2 : \mathbb{R}^m \to \mathbb{R}^n$ surjects, then what is the nullity of T_2?

   Solution. In order to surject, the rank has to be 2. By the rank-nullity theorem, the nullity is 1; since the rank plus the nullity has to be three. In the general case, the rank has to be n to surject, and then the nullity has to be $m - n$ so that they sum up to m. (Remember that in order for T_2 to surject it has to at least be true that $m \geq n$.)

3. Let A, x_1, x_2, y be as given below. Find a basis for the kernel of A. Next, note that $Ax_1 = y$ and $Ax_2 = y$. Check that $x_1 - x_2$ is in the kernel of A two different ways: first compute $A(x_1 - x_2)$, and second show that $x_1 - x_2$ is a linear combination of the basis you computed for the kernel of A.

   Solution. I get that the rref of A is the matrix

$$\begin{bmatrix} 1 & 0 & 4 & 0 & -4/3 \\ 0 & 1 & 1 & 0 & -2/3 \\ 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   A basis for the kernel then is $z_1 := (-4, -1, 1, 0, 0), z_2 := (4, 2, 0, -1, 3)$ using the usual method of finding the solutions. Check explicitly that $A(x_1 - x_2) = \vec{0}$ by direct calculation. Then, check whether $x_1 - x_2$ is a linear combination of z_1, z_2. To do this, I consider the matrix whose columns are $z_1, z_2, x_1 - x_2$, and find its rref:

$$\begin{bmatrix} -4 & 4 & -4 \\ -1 & 2 & -3 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We find that $x_1 - x_2 = -z_1 - 2z_2$, and hence it is in the kernel.