Math 110 — Midterm Exam — OCTOBER 2004

INSTRUCTIONS: The exam has four sets of questions labeled **I** through **IV**.

Most of the questions ask for some sort of explanation or justification for your answers. These explanations aren't necessarily very long, and in fact most are quite short. What's important is that you clearly state what principles, ideas, or reasoning you are using to arrive at your conclusion.

Each question is worth the same mark, although they are not all of the same difficulty.

Name: ______________________________

Question	Mark	Out of
I.		25
II.		25
III.		25
IV.		25

Total:		100

I. — TRUE OR FALSE.

Clearly indicate which answer you are choosing. No explanation is required.

(a) If A is any matrix, and $B = \mathrm{RREF}(A)$, then $\mathrm{rank}(A) = \mathrm{rank}(B)$.
. . . . **True** ☐ **False** ☐

(b) The matrix for rotation by angle θ in $\mathbb{R}^2$ always has rank 2. . . **True** ☐ **False** ☐

(c) If $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^5$ is a linear transformation, and we know what T does to the vectors $(1, 2, 0)$, $(2, 0, 1)$, and $(3, 0, 0)$, then we know what T does to any vector in $\mathbb{R}^3$.
. . . . **True** ☐ **False** ☐

(d) If $n > m$ then a system of m equations with n unknowns always has a solution.
. . . . **True** ☐ **False** ☐

(e) If A is an invertible $n \times n$ matrix, and B is an invertible $n \times n$ matrix, then the product AB is an invertible $n \times n$ matrix. . . . . **True** ☐ **False** ☐

II. — The linear transformation $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ is given by the matrix

$$A = \begin{bmatrix} 1 & 4 & 24 & 3 & 49 \\ 5 & 0 & 20 & 1 & 37 \\ 7 & 1 & 33 & 1 & 60 \\ 0 & 2 & 10 & 0 & 18 \end{bmatrix} \text{ which has RREF } \begin{bmatrix} 1 & 0 & 4 & 0 & 7 \\ 0 & 1 & 5 & 0 & 9 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

something you can assume without justification.

(a) Find a basis for the image $\mathrm{im}(T)$ of T, and justify your answer.

(b) Find a basis for the kernel $\ker(T)$ of T, no explanation necessary.

(Question II continued . . .)

(c) Find all solutions to the system of linear equations

$$\begin{array}{rcrcrcrcrcl}
1x & + & 4y & + & 24z & + & 3u & + & 49v & = & 5 \\
5x & & & + & 20z & + & 1u & + & 37v & = & 5 \\
7x & + & 1y & + & 33z & + & 1u & + & 60v & = & 8 \\
 & & 2y & + & 10z & & & + & 18v & = & 2
\end{array}$$

and briefly explain why these are all the solutions.

III. — Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation. Explain carefully why there is an $m \times n$ matrix A such that, for every vector $\vec{x}$ in $\mathbb{R}^n$, $T(\vec{x})$ is the same vector in $\mathbb{R}^m$ as $A\vec{x}$.

The issues to cover are: How you find the matrix A?, and how you know that $T(\vec{x}) = A\vec{x}$ for every vector $\vec{x}$ in $\mathbb{R}^n$, and not just some special ones?

IV. — Consider these three vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

I want a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ so that

$$T(\vec{v}_1) = \vec{v}_2, \quad T(\vec{v}_2) = \vec{v}_3, \quad \text{ and } T(\vec{v}_3) = \vec{v}_1.$$

(a) Find the 3×3 matrix A for this linear transformation.

(Question IV continued ...)

(b) Compute A^2.

(c) Compute A^3. Thinking of how T was defined, does this make sense?

(d) What is the inverse matrix for A?

(End of Exam)