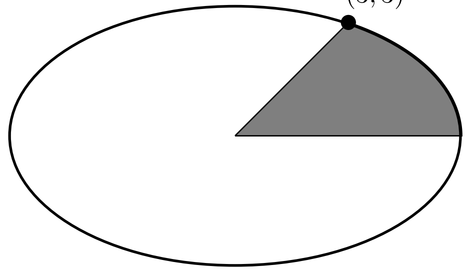
1. The ellipse at right is given by the equation

$$\frac{x^2}{6^2} + \frac{y^2}{12} = 1,$$

and the point marked is the point $(3, 3)$. Find the area of the shaded region (which connects the point, the origin, and $(6, 0)$).



HINT: The ellipse is the image of the unit circle under a linear transformation. First, figure out what that linear transformation is. Then figure out what region in the unit circle is sent to the shaded area under that linear transformation. If you can compute the area of the region in the unit circle, you'll be able to compute the area of its image in the ellipse.

2. Suppose that $f : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a multilinear function (but not necessarily alternating, or anything like that), and that we know the following eight values of f:

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) = e, \qquad f\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = \sqrt{7}$$

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) = 0 \qquad f\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = 2,$$

$$f\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) = \sqrt{5}, \quad f\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = 0,$$

$$f\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) = \pi, \qquad f\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = 3.$$

Compute these values of f:

(a) $f\left(\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}7\\4\end{bmatrix}\right)$ (b) $f\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}1\\5\end{bmatrix}\right)$

(c) $f\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right)$ (d) $f\left(\begin{bmatrix}-2\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}5\\7\end{bmatrix}\right)$

Leave your answers in symbolic form, i.e., if your answer is $4\pi + 6e - 3 + 2\sqrt{7} - 8\sqrt{5}$, leave it like that instead of writing $13.279020387244307976\ldots$.

3. In the definition of the general determinant, we saw that there was only *one* function $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which was multilinear, alternating, and took the value 1 when we plugged $\vec{e}_1$, $\vec{e}_2$, $\ldots$, $\vec{e}_n$ into the function.

In that argument, it's actually important that the function take n vectors in $\mathbb{R}^n$ as input. In this question we're going to see why by making the number of inputs different from the size of the vectors we plug in.

(a) First let's start by making the number of inputs one more than the size of the vector we plug in. Suppose that we have a function $f : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ which is multilinear (in this case, trilinear) and alternating. If $\vec{v}_1 = (a, b)$, $v_2 = (c, d)$ and $\vec{v}_3 = (e, g)$, compute the only possible value for $f(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. (Compute this by writing each vector $\vec{v}_i$ as a linear combination of $\vec{e}_1$ and $\vec{e}_2$ and expanding using the multilinear and alternating rules).

(b) Now let's make the number of inputs one less than the size of the vector we put in. Suppose that $f : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a function which is multilinear and alternating.

In the definition of the three dimensional determinant, the condition of being multilinear and alternating only needed one more piece of information (that $\det_3(\vec{e}_1, \vec{e}_2, \vec{e}_3) = 1$) to determine $\det_3$ completely.

In the case of the function f above, how many extra pieces of information do we need? For instance, would it be enough just to know the value of $f(\vec{e}_1, \vec{e}_2)$ in order to be able to determine $f(\vec{v}_1, \vec{v}_2)$ for all vectors $\vec{v}_1$, and $\vec{v}_2$ in $\mathbb{R}^3$? What *do* we need to know in order to be able to pin down the function f?

4. Suppose that $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ are vectors in $\mathbb{R}^3$, and we know that $\det_3(\vec{v}_1, \vec{v}_2, v_3) = 4$ and that $\det_3(\vec{v}_1, \vec{v}_4, \vec{v}_3) = 7$.

Compute:

(a) $\det_3(\vec{v}_4, \vec{v}_1, \vec{v}_3)$,

(b) $\det_3(\vec{v}_1, \vec{v}_2 + 3\vec{v}_4, \vec{v}_3)$,

(c) $\det_3(3\vec{v}_2,\ -2\vec{v}_1,\ \vec{v}_3)$,

(d) $\det_3(\vec{v}_1, \vec{v}_2 + 7\vec{v}_3, \vec{v}_3)$, and

(e) $\det_3(\vec{v}_1 + 4\vec{v}_2, \vec{v}_2 - 2\vec{v}_3, 6\vec{v}_1 + 4\vec{v}_3)$.