DUE DATE: MAR. 30, 2006

1. Let $T : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ be the map given by

$$T(f) = f'' - 2f' - 3f$$

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(a) Show that T is a linear transformation.

(b) Find a linear combination $a\sin(x) + b\cos(x)$ which solves the equation

$$f'' - 2f' - 3f = 2\cos(x).$$

(c) Check that e^{-x} and e^{3x} are in $\ker(T)$.

(d) Find a solution to

$$f'' - 2f' - 3f = 2\cos(x)$$

which also satisfies $f(0) = 2$ and $f'(0) = 3$.

2. Suppose that V is a vector space over $\mathbb{R}$ (*not* necessarily finite dimensional), and that $T_1 : V \longrightarrow V$ and $T_2 : V \longrightarrow V$ are linear transformations from V to V with the property that $T_3 = T_2 \circ T_1$ is the identity transformation, i.e. that $T_3(v) = v$ for all vectors v in V.

(a) Prove that T_1 is injective.

(b) Prove that T_2 is surjective.

Now suppose that $V = \mathbb{R}^\infty$, that T_1 is the "shift to the right" linear transformation from homework 9, #3(a), and that T_2 is the "shift to the left" linear transformation given by

$$T(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_3, x_4, \ldots)$$

(c) Check that $T_2 \circ T_1$ is the identity transformation from V to V.

(d) Show that T_1 is not surjective.

(e) Show that T_2 is not injective, and find its kernel.

Now suppose that V is a finite dimensional vector space, and that T_1 and T_2 are linear transformations from V to V like above (with $T_2 \circ T_1$ the identity transformation).

(f) Show that T_1 must be surjective.

(g) Show that T_2 must be injective.

What made the difference in the finite dimensional case?

3. Suppose that V and W are vector spaces and that $T : V \longrightarrow W$ is an isomorphism. If $v_1, \ldots, v_k$ is a basis for V then prove that $T(v_1), T(v_2), \ldots, T(v_k)$ is a basis for W.

4. Let B be a 4×4 diagonal matrix, and let W_B be the set of 4×4 matrices which commute with B (i.e., those matrices A so that $AB = BA$).

(a) Check that W_B is a a subspace of $M_{4\times 4}(\mathbb{R})$.

(b) What is $\dim(W_B)$?

Note that the answer to part (b) depends on which diagonal matrix B we start with, i.e., there is more than one possibility. For instance, if $B = I_4$, the 4×4 identity matrix, then all the 4×4 matrices commute with B, so $W_B = M_{4\times 4}(\mathbb{R})$ and the answer is 16.

If B is a different diagonal matrix, the dimension might also be different. The question is to find what the possible dimensions of W_B are, and explain how to figure out the dimension just by looking at B.