1. One model for a falling body, taking wind resistance into account, is given by the differential equation

$$m\frac{dV}{dt} = mg - kV.$$

where V is the velocity at time t. We've been using f for the name of the functions, and x for the name of the variable; dividing both sides by m and rearranging a bit, this is how the equation looks in the notation we've been using:

$$f' + \frac{k}{m}f = g$$

Find the general solution to this differential equation. The symbols k, m, and g are positive numbers. Also, sketch a few solutions to this equation, and try and explain what the behaviour of the solutions means physically.

2. Let $V = C^\infty(\mathbb{R})$, let T_1 be the linear transformation $T_1 : V \longrightarrow V$ given by

$$T_1(f) = f' - f,$$

and let $T_2 : V \longrightarrow V$ be the linear transformation given by

$$T_2(f) = f' + 2f.$$

(a) Compute $T_2 \circ T_1$ by describing where it sends a function f.

(b) Compute $T_1 \circ T_2$ by describing where it sends a function f.

By what we proved in class, these two compositions must be the same. But now let

$$T_3(f) = \sin(x)f' - f, \text{ and } T_4(f) = f' + 2xf.$$

Both of these are linear transformations from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$.

(c) Compute $T_4 \circ T_3$ by describing where it sends a function f.

(d) Compute $T_3 \circ T_4$ by describing where it sends a function f.

In this case, these aren't the same. The composition is independent of the order only when the differential operator has constant coefficients (instead of functions as coefficients, as in T_3 and T_4).

3. Physical systems often give rise to *coupled* differential equations, that is, equations involving more than one function, but where the derivatives of each function depend on all the other functions. An example of such a problem is trying to find functions f and g which solve the differential equations

$$\begin{aligned} f' &= f - 2g \\ g' &= 3f + 6g \end{aligned}$$

Subject to the conditions that $f(0) = 3$ and $g(0) = 5$.

Let $h_1 = f + g$ and $h_2 = 3f + 2g$. If f and g are solutions to the above problem,

(a) What must $h_1(0)$ and $h_2(0)$ be?

(b) Use the equations above to work out h_1' and h_2'. You will be able to write the derivative of h_1 in terms of just h_1 and the derivative of h_2 in terms of just h_2.

(c) Solve the conditions in (a) and (b) to find the functions h_1 and h_2.

(d) Now use the answer from (c) to find the functions f and g solving the system of differential equations and the conditions on f and g..

(e) To see where the mystery linear combinations h_1 and h_2 came from, find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}.$$

4. Find the general solutions to

(a) $f'' - f' - 2f = \sin(x)$.

(b) $f'' + 2f' - 15f = x^2$.

(c) $f'' + 5f' + 6f = x^3 - 2\cos(x)$.

The usual method of finding the general solution is to find one solution, and then add things in the kernel. In order to find the first solution, you can either decompose the appropriate differential operator as a composition of simple operators, and then use the explicit formula for solving the simpler operators given in class, or (a much easier method) try functions f "of the same type" as the answer you're looking for (e.g., for $\sin(x)$, try sin's and cos's) and use linearity.