1. Let $T_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear transformation given by the matrix

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix}$$

and let $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^1$ be the linear transformation given by the matrix $\begin{bmatrix} 5 & 3 \end{bmatrix}$.

(a) The kernel of T_1 is one dimensional. Find a vector u_1 which spans $\ker(T_1)$ (i.e., a basis for $\ker(T_1)$).

(b) The kernel of T_2 is also one-dimensional. Find a vector v_1 which is a basis for $\ker(T_2)$.

(c) Find a vector w_1 in $\mathbb{R}^3$ with $T_1(w_1) = v_1$.

(d) Check that u_1, w_1 is a basis for $\ker(T)$, where $T = T_2 \circ T_1$.

[This question is just a concrete example of the main idea used to prove the lemma from Tuesday's class]

2. Suppose that $T : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ is a linear constant coefficient differential operator (i.e. the things we've been looking at for two weeks) given by

$$T(f) = f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f' + a_0 f,$$

where the a_i's are real numbers and $a_0 \neq 0$. If we're trying to solve the equation $T(f) = g$ for some function g in $C^\infty(\mathbb{R})$, one way to try and find it is to write T as a composition $T = T_1 \circ \cdots \circ T_n$, and, using the explicit formula for solving the T_i's, work our way up the chain. This seems like a lot of work, and in the previous assignment it was suggested that an easier way might simply be to "try things that look like g" in T, and see if we can cook up a linear combination which works.

This doesn't sound like a very mathematical statement, but in fact we can be a lot more precise, and prove rigourously that that kind of idea will work.

In this problem, let's prove that if g is a polynomial, we can always find a polynomial f with $T(f) = g$.

Let d be any positive integer, and P_d the vector space of polynomials of degree $\leq d$. The vector space P_d is a subspace of $C^\infty(\mathbb{R})$.

(a) If f is in P_d (and so also in $C^\infty(\mathbb{R})$) explain why $T(f)$ is also in the subspace P_d (instead of just being some arbitrary thing in $C^\infty(\mathbb{R})$ which might not be in P_d).

Since (by part (a)) every f in P_d gets sent back into P_d by T, we can also consider T as giving a linear transformation $T : P_d \longrightarrow P_d$.

(b) Show that this linear transformation from P_d to P_d must be injective. (HINT: Don't we already know everything about $\ker(T)$?)

(c) Explain why this means that the map from P_d to P_d must also be surjective.

(d) If g is a polynomial of degree $\leq d$, explain why we know that there is a polynomial f, also of degree $\leq d$ with $T(f) = g$.

If it makes it easier to explain your reasoning, you can assume that all the roots of the characteristic polynomial of T are distinct.

3. Find the general solution to the following differential equations

(a) $f'' - 4f' + 4f = -4 + 12x$.

(b) $f'' - 6f' + 9f = 20 - 12x + 9x^2$.

(c) $f^{(3)} - 3f'' + 3f' - f = 22 - 5x$.

REMINDER: If T is a constant coefficient linear differential operator whose characteristic polynomial factors as

$$(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_r)^{m_r}$$

Then a basis for $\ker(T)$ is

$$\begin{gathered}
e^{\lambda_1 x}, xe^{\lambda_1 x}, x^2 e^{\lambda_1 x}, \ldots, x^{m_1 - 1} e^{\lambda_1 x}, \\
e^{\lambda_2 x}, xe^{\lambda_2 x}, x^2 e^{\lambda_2 x}, \ldots, x^{m_2 - 1} e^{\lambda_2 x}, \\
\vdots \\
e^{\lambda_r x}, xe^{\lambda_r x}, x^2 e^{\lambda_r x}, \ldots, x^{m_r - 1} e^{\lambda_r x}.
\end{gathered}$$