1. Let A be the matrix

$$A = \begin{bmatrix} 3 & 2 & z \\ x & 1 & 3 \\ 2 & y & 5 \end{bmatrix},$$

where x, y, and z are variables.

(a) Compute the adjoint matrix of A (this will still involve the variables x, y, and z).

(b) Compute the product of the adjoint matrix and A.

(c) Compute $\det(A)$.

(d) Assuming that $\det(A) \neq 0$, write down the inverse of A (this will still be a matrix involving the variables).

(e) To show that this method really gives a "universal formula for the inverse", plug in the values $(x,y,z) = (3,-1,1)$ into both A and the inverse matrix from part (d), and multiply them to see that it really gives the inverse for A.

(f) Do the same for $(x,y,z) = (1,1,1)$

2. Suppose that A is an $n \times n$ matrix with integer entries, and that A is invertible. Since A is invertible, we can compute the matrix A^{-1}. The computations seem much cleaner (and friendlier) when A^{-1} also has only integer entries. The purpose of this question is to figure out when that can happen.

PROVE: That (if A is an invertible $n \times n$ matrix, with only integer entries) A^{-1} has integer entries if and only if $\det(A) = \pm 1$ (where "$= \pm 1$" means equals 1 or equals -1).

REMINDER: Since this is an "if and only if", don't forget that means that you have two directions to prove.

3. Suppose that x_1, x_2, ..., x_n are numbers. The $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ x_1^3 & x_2^3 & x_3^3 & \cdots & x_n^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$$

is called the *Vandermonde matrix*, and is surprisingly useful to know a few basic facts about it. Let's establish one of them now.

If any of the two x_i's are equal to each other, then of course $\det(A) = 0$ since we will have two repeated columns. What we'd like to show is that if all of the x_i's are different, then $\det(A) \neq 0$, i.e., A is an invertible matrix.

(a) Explain why showing that $\det(A) \neq 0$ is the same as showing that $\det(A^t) \neq 0$, where A^t means the transpose of A. [This is a very short answer].

(b) Explain why showing that $\det(A^t) \neq 0$ is the same as showing that A^t has no nonzero vector in the kernel.

(c) if $\vec{v} = (c_0, c_1, \ldots, c_{n-1})$ is a vector in the kernel of A^t, and $\vec{v}$ is not the zero vector, explain how this would give you a polynomial of degree $\leq n-1$ with more than $n-1$ roots, which would be a contradiction. [HINT: consider what $A^t\vec{v} = 0$ means. Make sure that you answer carefully, for instance, why is it important that all the x_i's be different?]

4. For each of the following two matrices, find the values of t for which the matrix is not invertible. For each of those values of t, find a basis for the kernel of the corresponding non-invertible matrix.

$$\text{(a)} \begin{bmatrix} t-12 & 15 \\ -6 & t+7 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} t-17 & 9 & -2 \\ 2t-35 & 19 & -4 \\ 2t+2 & -(t+1) & t-1 \end{bmatrix}.$$

5. For the basis $\mathfrak{B} = [(3,5),(1,2)]$ in $\mathbb{R}^2$,

(a) Express $(4,3)$, $(1,2)$, and $(1,3)$ in $\mathfrak{B}$ coordinates.

(b) Express $(2,0)_{\mathfrak{B}}$, $(1,1)_{\mathfrak{B}}$, and $(-1,4)_{\mathfrak{B}}$ in the standard coordinates.

Just for fun, here is a chart showing the (capital) German Gothic letters. Happy Translating!

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{D}$	$\mathfrak{E}$	$\mathfrak{F}$	$\mathfrak{G}$	$\mathfrak{H}$	$\mathfrak{I}$	$\mathfrak{J}$	$\mathfrak{K}$	$\mathfrak{L}$	$\mathfrak{M}$	$\mathfrak{N}$	$\mathfrak{O}$	$\mathfrak{P}$	$\mathfrak{Q}$	$\mathfrak{R}$	$\mathfrak{S}$	$\mathfrak{T}$	$\mathfrak{U}$	$\mathfrak{V}$	$\mathfrak{W}$	$\mathfrak{X}$	$\mathfrak{Y}$	$\mathfrak{Z}$