1. Suppose that $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is given (in the usual coordinates) by the matrix

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix}.$$

If $\mathfrak{B} = [(1,1,1),\ (2,0,1),\ (3,2,1)]$ is a new basis in $\mathbb{R}^3$, and $\mathfrak{A} = [(3,5),\ (1,2)]$ a new basis in $\mathbb{R}^2$, find the matrix for T with respect to the new basis on both sides.

2. Suppose that $\vec{v}_1 = (1,1)$, $\vec{v}_2 = (1,-1)$, and that the basis $\mathfrak{B}$ is $\mathfrak{B} = [\vec{v}_1, \vec{v}_2]$.

Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $T(\vec{v}_1) = \vec{v}_1$ and $T(\vec{v}_2) = \vec{0}$.

(a) Write down the matrix for T in the new basis $\mathfrak{B}$. (You should be able to do this directly from the definition of T).

(b) Use this to write down the matrix for T in the standard basis.

You might want to compare the answer for (b) with the answer for homework 4, question 4, term 1, with $m = 1$. Can you see why these are the same?

NOTE: In part (b) you need to go from the matrix in $\mathfrak{B}$-basis form to the matrix in standard basis form, which is the reverse of what we did in class, so think for a bit to figure out which way the change of basis matrices should go.

3. We'll check in class on Tuesday that the determinant of a square matrix doesn't change when we change basis. The purpose of this question is to show that the *trace* of a matrix also doesn't change when we change basis.

For an $n \times n$ matrix A, the *trace* of A, $\mathrm{tr}(A)$ is the sum of the numbers on the diagonal. For instance, if

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 9 \\ 6 & 0 & 4 \end{bmatrix}$$

then $\mathrm{tr}(A) = 1 + 7 + 4 = 12$.

In the a_{ij} notation for the entries of a matrix, $\mathrm{tr}(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$.

(a) If A and B are $n \times n$ matrices, prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ (the formula for ij-th entry for a product of matrices on page 139 of the book may help).

(b) Suppose that A is an $n \times m$ matrix and B an $m \times n$ matrix. Then both products AB (an $n \times n$ matrix) and BA (an $m \times m$ matrix) are square matrices, so we can take their traces. PROVE OR DISPROVE: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ in this case.

(c) If C is an $n \times n$ matrix, and M an invertible $n \times n$ matrix, prove that

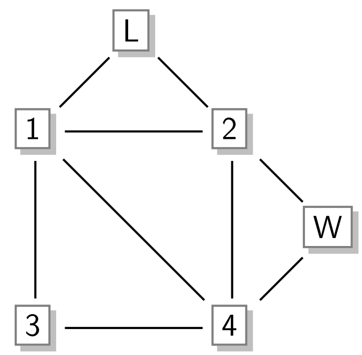
$$\mathrm{tr}(C) = \mathrm{tr}(M^{-1}CM).$$

SUGGESTION: If you make the right choice of matrices A and B, part (c) will follow from part (a) with very little work.

4. For each of the following matrices, find the eigenvalues, and for each eigenvalue find an eigenvector with that eigenvalue.

(a) $\begin{bmatrix} -18 & 8 \\ -60 & 26 \end{bmatrix}$ (b) $\begin{bmatrix} 27 & -48 \\ 16 & -29 \end{bmatrix}$ (c) $\begin{bmatrix} -3 & 2 & 7 \\ 0 & -44 & 60 \\ 0 & -36 & 49 \end{bmatrix}$

5. The diagram at right shows a simple Win/Lose game. In each turn, we are at one of positions 1, 2, 3, or 4, or we have either won (W) or lost (L). If we are at position W or L we stay there since the game is over. If we're at any of the positions 1 through 4, then during that turn we leave (with equal probability) along one of the lines coming out of that position. In other words, if we're at position 1, then there's a 1/4 chance that we'll end up at L the next turn, a 1/4 chance of ending up at 2, a 1/4 chance of ending up at 3, and a 1/4 chance of ending up at 4.



On the other hand, if we're at position 3, then there's a 1/2 chance of ending up at position 1 and a 1/2 chance of ending up at position 4 on the next turn.

(a) Write down the 6×6 transition matrix that tells us how to get from one turn of the game to another. When writing down the matrix, let's use the order of positions 1, 2, 3, 4, W, and then L.

If you write down the correct matrix, the eigenvectors should be: $\vec{v}_1 = (1, -1, -1, 1, 0, 0)$, $\vec{v}_2 = (3, 2, 2, 3, -5, -5)$, $\vec{v}_3 = (0, -4, 2, 0, 1, 1)$, $\vec{v}_4 = (5, 0, 0, -5, 1, -1)$, $\vec{v}_5 = (0, 0, 0, 0, 1, 0)$, and $\vec{v}_6 = (0, 0, 0, 0, 0, 1)$.

(b) Suppose we get to choose which of the squares 1, 2, 3, or 4 we start on. Which one gives us the better chance of winning?

(c) What is the chance of winning if we start on that square? Explain your answer.