DUE DATE: FEB. 16, 2006

1. The FIBONACCI NUMBERS are the numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1}+F_{n-2}$ for $n \geq 2$. (So for instance $F_2 = F_1+F_0 = 1+0 = 1$, $F_3 = F_2+F_1 = 1+1 = 2$, and $F_4 = F_3 + F_2 = 2 + 1 = 3$, etc.)

(a) Suppose that we set $\vec{w}_n = (F_n, F_{n-1})$ for $n \geq 1$, and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Show that the recursion relation above means that $\vec{w}_{n+1} = A\vec{w}_n$.

(b) Use the eigenvectors of A to find a formula for $A^k\vec{w}_1$ for any $k \geq 0$.

(c) Use the answer from (b) to find a formula for the n-th Fibonacci number F_n.

(d) The LUCAS NUMBERS are the numbers defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Find a formula for the n-th Lucas number.

2. For the following three matrices, find their characteristic polynomials, the algebraic and geometric multiplicities of each eigenvalue, and a basis for each of their eigenspaces.

(a) $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 & -2 \\ -1 & -1 & 1 \\ -1 & -3 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} -6 & 9 & 6 \\ 0 & 3 & 0 \\ -12 & 12 & 11 \end{bmatrix}$

To make factoring the characteristic polynomials a bit easier, 2 is a root of each one.

(d) Which of the matrices above is diagonalizable?

(e) For each of the matrices A from part (d), find an invertible matrix N so that $N^{-1}AN$ is a diagonal matrix.

3. Let A be the matrix

$$A = \begin{bmatrix} 19 & -14 \\ 21 & -16 \end{bmatrix}.$$

Find a formula for the entries of A^k (for $k \geq 1$). As a check, compute A^2 and A^3 and to see if they match your formulas.

HINT: The first column of any 2×2 matrix is "where $(1, 0)$ gets sent", therefore if $\vec{w}_0 = (1, 0)$, then the first column of A^k is just $A^k\vec{w}_0$.

4. Suppose that A_n is the $n \times n$ matrix which has 2's on the diagonal, and 1's everywhere else:

$$A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \; A_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \; A_4 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}, \ldots$$

and suppose that B_n is the $n \times n$ matrix which is just filled with minus-ones:

$$B_2 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \; B_3 = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \; B_4 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}, \ldots$$

I'd like to find a formula for $\det(A_n)$.

(a) Explain why $\det(A_n) = \det(I_n - B_n)$, where I_n is the $n \times n$ identity matrix.

(b) If $P_n(\lambda)$ is the characteristic polynomial of B_n, explain why $\det(A_n) = P_n(1)$.

This means that we can compute $\det(A_n)$ by first figuring out the characteristic polynomial of B_n and then plugging in a value. It might seem like more work to compute the characteristic polynomial of B_n, but

(c) Since B_n has rank 1, explain why this means that λ^{n-1} has to divide $P_n(\lambda)$. (HINT: How big is the kernel of B_n? What is the relation between the kernel of B_n and the eigenspace E_0 for B_n?).

This means that $P_n(\lambda)$ is of the form $\lambda^{n-1}(\lambda - a)$ for some number a.

(d) Either by looking at the trace of B_n, or by seeing what happens to the vector $\vec{v} = (1, 1, \ldots, 1)$ of all 1's when you put it through B_n, find the value of a.

(e) What is $\det(A_n)$?

(f) What is the determinant of the $n \times n$ matrix C_n which has 5's on the diagonal, and 1's everywhere else?