1. These answers should be short. The matrix A is an $n \times n$ matrix.

(a) Explain why A has the same eigenvalues (with the same algebraic multiplicities) as A^t. (although it won't necessarily have the same eigenvectors).

(b) If N is an invertible matrix, explain why A as the same eigenvalues, with multiplicity, as $N^{-1}AN$ (although again it won't necessarily have the same eigenvectors).

For the next two questions, you can assume that there is a basis consisting of eigenvectors of A. Under this assumption,

(c) How do the eigenvalues of A compare with the eigenvalues of A^2?

(d) If A is also an invertible matrix, how do the eigenvalues of A compare with the eigenvalues of A^{-1}?

2. We say the $n \times n$ matrix A is *similar* to the $n \times n$ matrix B if there is an invertible $n \times n$ matrix N so that $B = N^{-1}AN$. Another way to think about this is that B is simply the matrix A after the change of basis given by the column vectors of N.

(a) Show that if A is similar to B, then B is similar to A. (i.e., if there is an invertible matrix N so that $B = N^{-1}AN$, which is what "A is similar to B" means, then there must also be some invertible matrix M so that $A = M^{-1}BM$, which is what "B is similar to A" means).

(b) Show that if A is similar to B, and B is similar to C, then A is similar to C. (i.e., if there is some invertible matrix N_1 with $B = N_1^{-1}AN_1$, and some invertible matrix N_2 with $C = N_2^{-1}BN_2$ then there is some invertible matrix N_3 with $A = N_3^{-1}CN_3$.)

The above properties should hopefully make sense: Part (a) says that if matrix B is matrix A after a change of coordinates, then matrix B is also matrix A after a change of coordinates (presumably, just the reverse change of cordinates...). Part (b) says that if matrix B is matrix A after a change of coordinates, and if matrix C is matrix B after another change of coordinates, then matrix C is also matrix A after a (single) change of coordinates.

(c) Suppose that A is an $n \times n$ matrix, and all the roots of its characteristic polynomial are distinct real numbers (i.e., all roots are real, and have multiplicity one). Explain why A is similar to a diagonal matrix. You don't have to prove all the steps, just give the reasons, based on what we know, why this is true.

(d) Suppose that A and B are two matrices with the same characteristic polynomial, and that characteristic polynomial has distinct real roots (i.e., like above, all roots are real and have multiplicity one). Explain why A must be similar to B.

(e) Find an example of 2×2 matrices A and B which have the same characteristic polynomial, but are not similar. (For instance, perhaps one is diagonalizable, and one is not.)

3. Suppose that A and B are $n \times n$ matrices, and that A is similar to B (as in question 2).

(a) Show that A^2 is similar to B^2 (Hint: just multiply.)

(b) Suppose that D is a diagonal matrix with real entries. If we want to find a diagonal matrix C with real entries, such that $C^2 = D$, what has to be true about the eigenvalues of D?

(c) Let $A = \begin{bmatrix} -16 & -10 \\ 50 & 29 \end{bmatrix}$.

Find a real matrix B with $B^2 = A$ (i.e., a "square root" of A).