

1. Put the following complex numbers in the form  $a + bi$ :

(a)  $(4 - 5i) - (6 - 3i)$  (b)  $(2 - 3i) \cdot (4 + 5i)$  (c)  $\frac{3 - 4i}{6 + 8i}$  (d)  $(1 + i)^{20}$

2. Find all the solutions to  $z^6 = 1$ , where  $z$  is a complex number. Draw the solutions on the unit circle in  $\mathbb{C}$ . Is there one solution whose powers give you all the rest? Is there more than one?

3. You can work out  $\sin$  and  $\cos$  of the angles  $\pi/4$  and  $\pi/3$  (or  $\pi/6$ ) by looking at some special triangles, and most people know these values. But, there are actually more values that can be worked out exactly. In this question, let's work out  $\sin(2\pi/5)$  and  $\cos(2\pi/5)$ .

To make the notation easier, let  $c = \cos(2\pi/5)$  and  $s = \sin(2\pi/5)$ . Let  $z = c + s \cdot i$ .

- (a) Explain why  $z^5 = 1$ .
- (b) Expand  $(c + s \cdot i)^5$  and collect the parts with  $i$  (the *imaginary part*). Explain why this will be zero, so to find values of  $c$  and  $s$ , we can start by looking for values where this polynomial in  $c$  and  $s$  is zero.

In fact, since  $s$  isn't zero, and since the above expression is divisible by  $s$ , we can divide it by  $s$  to get a degree four polynomial in  $c$  and  $s$ .

- (c) Explain why  $s^2 = 1 - c^2$ , and substitute it into the polynomial above to get a polynomial of degree 4 which only involves the variable  $c$ .
- (d) If you look more closely, the polynomial from part (c) can also be considered as a polynomial in  $c^2$ , of degree 2 (i.e., you can write everything in terms of  $c^2$ ). Use the quadratic formula to solve for  $c^2$ , and then take square roots to find the four possible solutions in  $c$  to this polynomial.
- (e) To decide which of the four values above is really  $\cos(2\pi/5)$ , draw a picture of  $z$  on the unit circle, and also draw a picture of the point at the angle  $\pi/4$ . From the picture, explain how you know that  $0 < \cos(2\pi/5) < 1/\sqrt{2}$ . There is only one solution to the equation in (d) which satisfies this conditions, so that must be  $\cos(2\pi/5)$ . What is it?
- (f) Use the formula  $s^2 = 1 - c^2$  to find  $\sin(2\pi/5)$ .

4. Let  $A$  be the matrix  $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ .

- (a) Compute  $A^2$ ,  $A^3$ ,  $A^4$ ,  $A^5$ , and  $A^6$ .
- (b) What will  $A^7$ ,  $A^8$ , and  $A^9$  be?
- (c) If we compute  $A^k$  for  $k = 1, 2, 3, 4, \dots$ , how many different matrices will we get?
- (d) Find the characteristic polynomial of  $A$ , and compute its (complex!) eigenvalues.
- (e) Looking at the answers to question (2), explain why  $A$  has the behaviour above.

[Thinking about the fact that there is a formula for the entries of  $A^k$  in terms of the eigenvalues of  $A$ , or what  $A$  would look like in diagonal form is one way to think about this, but there are others.]