1. In the fish population problem from Thursday's class, the matrix representing the transition from one year's population to the next was

$$\begin{bmatrix} 0.7 & 0.2 \\ 3 & 0 \end{bmatrix}$$

representing a 70% adult survival rate from year to year, a 20% survival rate for young fish, and the fact that an adult fish produces on average 3 young fish per year.

With these numbers, the population grows at an asymptotic rate of 20% (i.e., a factor of 1.2) per year. Suppose that we decide to allow fishing, and allow a yearly fraction of f adult fish to be caught (and we arrange fishing season so that it will not interfere with the reproduction of the fish). The new matrix describing this situation is:

$$A = \begin{bmatrix} 0.7 - f & 0.2 \\ 3 & 0 \end{bmatrix}$$

If we pick f very small (close to zero) the population will still grow. If we pick f too large, the population will die off. We want to pick f "just right" so that the population will be stable.

(a) What does this stability condition mean in terms of the largest (i.e., dominant) eigenvalue of A?

(b) What does f have to be in order for this to happen?

(c) For this value of f, find the eigenvalues and eigenvectors of A.

(d) If initially there are no fish in our lake, and we wish to stock it with adult fish (and no young fish) so that the total number of fish is $10,000$ when it hits the stable level (counting both adult and young fish together, and with the f from above), how many adult fish should we introduce into the lake to make this happen?

2. Draw the Gerschgorin discs for the following matrices

$$\text{(a)} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} i & 0 & i \\ 0 & -i & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} 3i & i & -1 \\ -1 & 4 & 1 \\ 2 & -1 & -4+i \end{bmatrix}$$

For the matrices in (a) and (b), find the eigenvalues and include them in the pictures.

3. Let A be an $n \times n$ matrix, whose entries could be real or complex numbers. Suppose that the diagonal entries of A are much larger than the size of the other entries in the same row, in the sense that for each row i

$$2\|a_{ii}\| > \|a_{i1}\| + \|a_{i2}\| + \|a_{i3}\| + \cdots + \|a_{in}\|.$$

Prove that A is invertible. (HINT: How is A *not* being invertible connected to the eigenvalues of A?) The matrix in 2(d) satisfies this condition, if you want an example to look at.

4. In class it was claimed that if we start with a "sufficiently random" vector $\vec{w}$, and repeat the procedure

"Compute $A\vec{w}$, and then scale the answer so that one of the coordinates is 1."

then the result (after repeating this a bunch of times) would converge on an eigenvector for the dominant eigenvalue of A.

The purpose of this question is to figure out what "sufficiently random" means, and why it is there. We'll use the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 2 \\ -1 & -3 & 5 \end{bmatrix}.$$

(a) Starting with the vector $\vec{w}_1 = (1, 0, 0)$, do this procedure 10 times (you don't need to show all the calculations, just show the first two and the final answer. You should re-scale so that the last coordinate is 1).

(b) What vector $\vec{v}_1$ does this seem to be approaching?

(c) Check that this vector from (b) *is* an eigenvector of A and find its eigenvalue.

(d) Find the eigenvalues for A (all are integers). What is the dominant eigenvalue? Does this seem to contradict the statement above and part (c)?

(e) Starting with the vector $\vec{w}_2 = (2, 0, 1)$, again repeat the procedure 10 times (again you don't need to show all the work).

(f) What vector $\vec{v}_2$ does this seem to be approaching? Check that it is an eigenvector of A and find the corresponding eigenvalue.

(g) Find all the eigenvectors of A, and write $\vec{w}_1$ and $\vec{w}_2$ as linear combinations of those eigenvectors.

(h) Use the answer for (g) to explain what happened in (c), and why things worked in (f).

(i) If we have a diagonalizable $n \times n$ matrix A, and we start with a vector $\vec{w}$ and repeat the procedure above, what condition on $\vec{w}$ will ensure that it converges to an eigenvector for the dominant eigenvalue of A? Will this condition be true for "most" vectors $\vec{w}$ in $\mathbb{R}^n$?