1. In each of the following cases, either prove that the given subset W is a subspace of V, or show why it is not a subspace of V.

(a) $V = C^\infty(\mathbb{R})$, W is the set of functions $f \in V$ for which $\lim_{x \longrightarrow \infty} f(x) = 0$.

(b) $V = C^\infty(\mathbb{R})$, W is the set of functions $f \in V$ for which $f(4) = 1$.

(c) $V = M_{2\times 2}(\mathbb{R})$, W is the set of matrices whose square is the zero matrix (i.e., those matrices A with A^2 the zero matrix).

(d) $V = \mathbb{R}^\infty$, W is the subset of vectors $(x_1, x_2, x_3, \ldots)$ for which $x_3 = x_2 + x_1$, $x_4 = x_3 + x_2$, $x_5 = x_4 + x_3$, $\ldots$, and in general $x_{n+2} = x_{n+1} + x_n$ for all $n \geq 1$.

2. In each of the following cases, either prove that the vectors are linearly independent, or show that they are linearly dependent.

(a) The vectors $v_1 = (1, 2)$ and $v_2 = (3, 6)$ in W_2.

(b) The vectors $v_1 = \ln(x^2 + 1)$, $v_2 = \ln(x^4 + 4x^2 + 3)$, and $v_3 = \ln(x^2 + 3)$ in $C^\infty(\mathbb{R})$.

(c) The vectors $v_1 = e^x$, $v_2 = e^{3x}$, and $v_3 = \cos(x)$ in $C^\infty(\mathbb{R})$.

3. In each of the following cases, either prove that the given rule is a linear transformation, or show why it isn't.

(a) $T : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$, T is the rule "shift to the right": $T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$.

(b) $T : M_{2\times 3}(\mathbb{R}) \longrightarrow \mathbb{R}^2$, T sends the matrix M to $M\vec{w}$ where $\vec{w} = (1, 2, 3)$ (the output is a vector in $\mathbb{R}^2$).

(c) $T : W_2 \longrightarrow \mathbb{R}^2$, $T(x, y) = (x, y)$.

(d) $T : W_2 \longrightarrow \mathbb{R}^2$, $T(x, y) = (\ln(x), \ln(y))$.

(e) $T : C^\infty(\mathbb{R}) \longrightarrow \mathbb{R}$, $T(f) = \displaystyle\int_1^3 f \cdot \sin(x)\, dx$.

The vector spaces used on the previous page are

$C^\infty(\mathbb{R})$: The vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ with infinitely many derivatives. Addition and scalar multiplication are just addition and scalar multiplication of functions.

$\mathbb{R}^\infty$: The set of infinite sequences $(x_1, x_2, x_3, \ldots)$ of numbers in $\mathbb{R}$, where addition and scalar multiplication are coordinate-wise.

$M_{m\times n}(\mathbb{R})$: The set of $m \times n$ matrices with entries in $\mathbb{R}$, where addition is addition of matrices and scalar multiplication is multiplying all entries of the matrix by that number.

W_2: The following "weird" vector space: W_2 is the set of all pairs (x, y) with both $x > 0$ and $y > 0$ real numbers, with addition defined as multiplication of the coordinates:

$$(x_1, y_1) + (x_2, y_2) = (x_1x_2, y_1y_1)$$

for any (x_1, y_1), (x_2, y_2) in W_2, and scalar multiplication defined as exponentiation of the coordinates:

$$c \cdot (x, y) = (x^c, y^c)$$

for any (x, y) in W_2 and c in $\mathbb{R}$.

REMINDER: The zero vector in W_2 is $(1, 1)$.

All of these vector spaces are vector spaces over $\mathbb{R}$.