Material Covered

To every linear map T from $\mathbb{R}^n$ to $\mathbb{R}^n$, we can find a constant p such that, for any parallelepiped S, $Area(T(S)) = p(Area(S))$. This p controls how the linear map transforms the areas of different parallelepipeds. We call this area transformation factor the *determinant* of T, denoted by $det(T)$. $det(T)$ can be positive or negative, depending upon whether T reflects the parallelepiped or just reorients it.

If T sends the standard basis $e_1, e_2, e_3, \cdots, e_n$ to the vectors $v_1, v_2, v_3, \cdots, v_n$, we define $det_n(v_1, \cdots, v_n)$ to be $det(T)$. $det_n(v_1, \cdots, v_n)$ is the signed area of the parallelepiped spanned by $v_1, \cdots, v_n$.

We know that the determinant is the unique function that satisfies these 3 properties:

1. It is n-linear. This means that you can pull plusses and constants out of the function: $det_n(v_1 + u_1, v_2, \cdots, v_n) = det_n(v_1, v_2, \cdots, v_n) + det_n(u_1, v_2, \cdots, v_n)$ and $det_n(a \times v_1, v_2, \cdots, v_n) = a \times det_n(v_1, v_1, \cdots, v_n)$. This property is called n-linear because it holds for each of the n arguments of the function det_n.

2. It is alternating. This means that if you switch two of the arguments, you multiply the function by -1: $det_n(v_1, v_2, \cdots, v_n) = -det_n(v_2, v_1, \cdots, v_n)$.

3. It gives the area 1 to the standard basis vectors $e_1, e_2, \cdots, e_n$.

We have many ways to calculate the determinant:

1. Use the multi-linearity of det_n to express the determinant. For example:
$$det_3\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = 2det_3\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right) + 2det_3\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

2. Use the "diagonal rule" for 2×2 and 3×3 matrices. Take the product of each up-left to down-right diagonal, and subtract the product of each up-right to down-left diagonal. In the case of 2×2 matrices we get the formula $ad - bc$.

3. Use the Laplace expansion. For an $n \times n$ matrix, pick one row or column. Then, for each entry in that row or column, remove that entry's row and column from the matrix to get a smaller matrix, take the determinant of that matrix, multiply by the entry in question, and multiply by ± 1 depending upon where that entry is in the matrix. Sum over all entries.

Practice Problems

1. Find a formula for the area of the triangle in $\mathbb{R}^2$ with vectors v_1 and v_2 as sides.

2. Find a formula for the volume of the "pyramid" in $\mathbb{R}^3$ with vectors v_1, v_2, and v_3 as edges. (Hint: the volume of the pyramid with e_1, e_2, and e_3 as edges is 1/6).

3. Suppose that function f from $\mathbb{R}^2$ to $\mathbb{R}$ is bilinear. Expand $f(v_1, v_2 + v_3)$, $f(2v_1, v_2)$, and $f(4v_1 + v_2, 3v_3)$.

4. Suppose that the function g from $\mathbb{R}^3$ to $\mathbb{R}$ is trilinear. Expand $g(3v_1, v_2, 5v_3)$, $g(v_1, v_2 + 2v_3, v_4)$, and $g(v_1 + 2v_2 + 3v_3, 4v_4 + v_5, v_6)$.

5. Let h from $\mathbb{R}^4$ to $\mathbb{R}$ be alternating, and set $a = h(v_1, v_2, v_3, v_4)$. Find $h(v_1, v_3, v_2, v_4)$, $h(v_3, v_2, v_1, v_4)$, $h(v_3, v_1, v_2, v_4)$, and $h(v_2, v_4, v_1, v_3)$ in terms of a.

6. Let v_1, v_2 and v_3 in $\mathbb{R}^3$ be such that $det_3(v_1, v_2, v_3) = 2$. Compute $det_3(v_2, 1/2v_1, v_3)$, $det_3(v_1 + v_2, v_1 - v_2, v_3)$, and $det_3(v_2, v_1 + 3v_3, v_3 + 3v_1)$.

7. Compute the determinants of the following matrices: $\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 & 1 \\ -4 & 1/2 & 2 \\ 6 & 3 & -1 \end{bmatrix}$,
$\begin{bmatrix} 1 & 1 & 3 & 1/5 \\ 2 & 2 & 1 & 4 \\ 1/3 & 1/3 & 2 & -2 \\ 0 & 0 & 3 & -1/2 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 & 1 & 0 & -2 \\ 1 & 1 & 0 & 0 & 3 \\ 5 & 0 & 0 & 0 & 0 \\ 1 & 2 & -3 & 1/5 & 3 \\ 0 & 4 & 2 & 0 & 2 \end{bmatrix}$

8. Compute $det_3(v_1, v_2, 3v_1 - 2v_2)$.

9. Find the flaw in the following argument:

   We wish to find the volume of the diamond with vertices at $(0, 0)$, $(2, 0)$, $(1, 3)$ and $(1, -3)$. We divide the diamond into two pieces. The top piece is a triangle with one side being the vector $v_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and another side being the vector $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Thus, it has an area equal to $1/2det_2(v_1, v_2)$. The bottom piece is a triangle with one side being the vector v_1, and another side being the vector $v_3 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Thus, it has an area equal to $1/2det_2(v_1, v_3)$. So, the total area of the diamond is $1/2det_2(v_1, v_2) + 1/2det_2(v_1, v_3)$, which equals $1/2det_2(v_1, v_2 + v_3)$ by bilinearity. But, $v_2 + v_3 = v_1$, so the total area is $1/2det_2(v_1, v_1) = 0$. Thus, the diamond has area 0.

Solutions

1. Area $= \frac{1}{2}|det_2(v_1, v_2)|$

2. Volume $= \frac{1}{6}|det_3(v_1, v_2, v_3)|$

3. $f(v_1, v_2 + v_3) = f(v_1, v_2) + f(v_1, v_3)$
   $f(2v_1, v_2) = 2f(v_1, v_2)$
   $f(4v_1 + v_2, 3v_3) = 12f(v_1, v_3) + 3f(v_2, v_3)$

4. $g(3v_1, v_2, 5v_3) = 15g(v_1, v_2, v_3)$
   $g(v_1, v_2 + 2v_3, v_4) = g(v_1, v_2, v_4) + 2g(v_1, v_3, v_4)$
   $g(v_1 + 2v_2 + 3v_3, 4v_4 + v_5, v_6) =$
   $4g(v_1, v_4, v_6) + 8g(v_2, v_4, v_6) + 12g(v_3, v_4, v_6) + g(v_1, v_5, v_6) + 2g(v_2, v_5, v_6) + 3g(v_3, v_5, v_6)$

5. $h(v_1, v_3, v_2, v_4) = -a$
   $h(v_3, v_2, v_1, v_4) = -a$
   $h(v_3, v_1, v_2, v_4) = a$
   $h(v_2, v_4, v_1, v_3) = -a$

6. $det_3(v_2, 1/2v_1, v_3) = -1/2 det_3(v_1, v_2, v_3) = -1$
   $det_3(v_1 + v_2, v_1 - v_2, v_3) = det_3(v_1, -v_2, v_3) + det_3(v_2, v_1, v_3) = -4$
   $det_3(v_2, v_1 + 3v_3, v_3 + 3v_1) = det_3(v_2, v_1, v_3) + 9det_3(v_2, v_3, v_1) = 16$

7. $\begin{vmatrix} 3 & 1 \\ -4 & 2 \end{vmatrix} = 10$

   $\begin{vmatrix} 2 & 3 & 1 \\ -4 & 1/2 & 2 \\ 6 & 3 & -1 \end{vmatrix} = -4$

   $\begin{vmatrix} 1 & 1 & 3 & 1/5 \\ 2 & 2 & 1 & 4 \\ 1/3 & 1/3 & 2 & -2 \\ 0 & 0 & 3 & -1/2 \end{vmatrix} = 0$

   $\begin{vmatrix} 2 & 3 & 1 & 0 & -2 \\ 1 & 1 & 0 & 0 & 3 \\ 5 & 0 & 0 & 0 & 0 \\ 1 & 2 & -3 & 1/5 & 3 \\ 0 & 4 & 2 & 0 & 2 \end{vmatrix} = -12$

8. $det_3(v_1, v_2, 3v_1 - 2v_2) = 0$