Material Covered

A constant coefficient linear differential equation is an equation of the form $f^{(n)}+a_{n-1}f^{(n-1)}+\cdots+a_1f'+a_0f=g$, where the a_i's are constants and g is some given function. The highest derivative of f which appears in the equation is called the degree of the equation. To solve such an equation, we must the solution set of all f which satisfy the equation. Such an equation can also be expressed in the form $T(f)=g$ where $T(f)$ is the linear map which sends f to $f^{(n)}+a_{n-1}f^{(n-1)}+\cdots+a_1f'+a_0f$.

To the linear map T we can associate a polynomial (the characteristic polynomial) which is equal to $\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0$. We can then factor the characteristic polynomial as $(\lambda-r_1)(\lambda-r_2)\cdots(\lambda-r_n)$ where the r_n's are the roots of the characteristic polynomial. Then, we can let T_i be the degree one linear map which sends f to $f'-r_if$, and then T can be decomposed as a composition of linear maps: $T=T_1\circ T_2\circ\cdots\circ T_{n-1}\circ T_n$. A composition of linear maps is the linear map that results from applying one map after the other. For example, $(T_1\circ T_2)(f)=T_1(T_2(f))$. By studying each of the maps in such a composition, we can discover many things about T.

We can see that a solution to an equation of the form $f'-rf=g$ will always be given by $f(x)=e^{rx}\int_0^x e^{-rt}g(t)dt$. This gives us a way to find a solution of $T_i(f)=g$ for each i. If we let f_1 be a solution of $T_1(f)=g$, f_2 be a solution of $T_2(f)=f_1$, etc. we will get f_n to be a solution of $T(f)=g$. While this method provides us a way of finding a solution to any differential equation of this form, it takes significantly longer than the "guess and check" method. Because $T_i(f)=g$ can always be solved, each T_i is surjective, and thus T is surjective. The surjectivity of T means that $T(f)=g$ can be solved for every g.

Once we have a single particular solution to the differential equation, we can find the general solution by adding to the particular solution every element of the kernel of T. We can calculate the kernel of T by looking at the kernels of each of the T_i's. We know that the kernel of $T_i(f)=f'-r_if$ is $\{ce^{r_ix}|c\in\mathbb{R}\}$, which is one-dimensional. The dimension of the kernel of a composition of surjective linear maps is the sum of the dimensions of the maps in the composition. This means that if T has degree n, it decomposes as a composition of n T_i's of degree one, and thus the kernel of T will be n-dimensional.

To calculate the kernel of T, we note that $x^de^{r_ix}$ is in the kernel of $(T_i)^k$ (T_i composed with itself k times) for all $d\leq k$. If $(\lambda-r_i)^k$ divides the characteristic polynomial, then we can pick the last k roots of the characteristic polynomial to all be r_i, and thus decompose T as $T=T'\circ(T_i)^k$. We can see that anything that is in the kernel of $(T_i)^k$ will be in the kernel of T, so for each distinct root, we can find k functions in the kernel of T, where k is the multiplicity of the roots. All these functions will be linearly independent, and there will be n of them (because the sum of the multiplicities of the roots of a polynomial is equal to its degree), so these functions will form a basis for the kernel of T. This gives us a way to calculate the kernel of T, and thus to solve the differential equation.

Practice Problems

1. For each pair of linear maps S and T, calculate $S \circ T$ and $T \circ S$.
   a) $S : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ sends f to $f'' - 2f' + 6f$, and $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ sends f to $f' + 3f$.
   b) $S : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ sends f to $\sin(x)f$, and $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ send f to f'.
   c) $S : R^2 \to R^2$ has matrix $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$, and $T : R^2 \to R^2$ has matrix $\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$.
   d) $S : P_n(\mathbb{R}) \to \mathbb{R}$ sends a polynomial to the sum of its coefficients, $T : \mathbb{R} \to P_n(\mathbb{R})$ sends a real number r to the polynomial rx^n.

2. Remember that two linear maps S and T are said to commute if $S \circ T = T \circ S$. Which of the pairs of maps in the previous question commute?

3. For each linear differential operator below, decompose it as a composition of degree one differential operators:
   a) T sends f to f''.
   b) T sends f to $f'' - f' - 2f$.
   c) T sends f to $f^{(4)} + 4f^{(3)} + 3f'' - 4f' - 4f$.

4. Compute the kernel of each of the following linear operators:
   a) T sends f to $f'' - f' - 2f$.
   b) T sends f to f''.
   c) T sends f to $f^{(4)} + 4f''' + 3f'' - 4f' - 4f$.

5. Find a particular solution to each of the following differential equations:
   a) $f'(x) - 5f(x) = \frac{1}{x}$.
   b) $f''(x) - f'(x) - 2f(x) = \sqrt{x}$.
   c) $f'''(x) - 2f''(x) + f'(x) + f(x) = x$.

6. Solve the differential equation $f'''(x) - 2f''(x) - 5f'(x) + 6f(x) = \sin x$.

7. What are the conditions on g such that the solution set to the differential equation $T(f) = g$ will be a subspace of $C^\infty(\mathbb{R})$? Why do these conditions guarantee that the solution set will be a subspace? Why are they necessary for it being a subspace? (HINT: every subspace must contain the zero vector)

8. Consider the linear map T which sends f to $f'' + f$.
   a) Decompose T as a composition $T_1 \circ T_2$.
   b) Based on T_1 and T_2 find functions f_1 and f_2 in the kernel of T.
   c) Confirm that $\sin x$ and $\cos x$ are in the kernel of T.
   d) Explain why f_1, f_2, sin and cos are linearly independent. Does this contradict the kernel being two-dimensional?
   e) Are f_1 and f_2 actually in $C^\infty(\mathbb{R})$? Are T_1 and T_2 actually linear maps from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$?

Solutions

1. For each pair of linear maps S and T, calculate $S \circ T$ and $T \circ S$.
   a) $S \circ T$ is the same as $T \circ S$; they both send f to $f''' + f'' + 18f$.
   b) $S \circ T$ sends f to $\sin(x)f'$, and $T \circ S$ sends f to $sin(x)f' + cos(x)f$.
   c) $S \circ T$ has matrix $\begin{bmatrix} -4 & 7 \\ -2 & 2 \end{bmatrix}$, and $T \circ S$ has matrix $\begin{bmatrix} -1 & 1 \\ -5 & -1 \end{bmatrix}$.
   d) $S \circ T$ sends a real number r to r, $T \circ S$ sends a polynomial P to the polynomial rx^n, where r is the sum of the coefficients of p.

2. Only the first pair of linear maps commute.

3. a) $T = T_1 \circ T_1$, where T_1 sends f to f'.
   b) $T = T_1 \circ T_1$, where T_1 sends f to $f' - 2f$ and T_2 sends f to $f' + f$.
   c) $T = T_1 \circ T_2 \circ T_3 \circ T_3$, where T_1 sends f to $f' - f$, T_2 sends f to $f' + f$, and T_3 sends f to $f' + 2f$.

4. a) The kernel has basis $\{e^{2x}, e^{-x}\}$.
   b) The kernel has basis $\{1, x\}$.
   c) The kernel has basis $\{e^x, e^{-x}, e^{-2x}, xe^{-2x}\}$..

5. a) $f(x) = e^{5x} \int_0^x \frac{e^{-5t}dt}{t}$.
   b) $f(x) = e^{2x} \int_0^x e^{-3y} \int_0^y e^t \sqrt{t} dt dy$.
   c) $f(x) = x - 1$.

6. The general solution is $f(x) = \frac{1}{13} \sin x + \frac{3}{26} \cos x + c_1 e^x + c_2 e^{-2x} + c_3 e^{3x}$.

7. The solution set to the differential equation $T(f) = g$ will be a subspace of $C^\infty(\mathbb{R})$? exactly when $g = 0$. When $g = 0$, this will be a subspace because it will be the kernel of T. If this is a subspace, 0 is in it, so 0 is a solution, and so $T(0) = g$, which means $g = 0$.

8. a) $T = T_1 \circ T_2$ where T_1 sends f to $f' - if$ and T_2 sends f to $f' + if$.
   b) $f_1(x) = e^{ix}$ and $f_2(x) = e^{-ix}$.
   c) $T(\sin x) = -\sin x + \sin x = 0$ and $T(\cos x) = -\cos x + \cos x$.
   d) f_1, f_2, sin and cos are linearly independent, because sin and cos are linearly independent, f_1 and f_2 are linearly independent, and we can't make f_1 and f_2 out of sin and cos because f_1 and f_2 have complex values, and sin and cos have real values.
   e) f_1 and f_2 are not actually in $C^\infty(\mathbb{R})$, because they don't have only real values?.T_1 and T_2 are not actually linear maps from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$, because they send functions from $\mathbb{R}$ to $\mathbb{R}$ to functions from $\mathbb{R}$ to $\mathbb{C}$.