Material Covered

The Laplace expansion method of calculating determinants. For an $n \times n$ matrix, pick one row or column. Then, for each entry in that row or column, remove that entry's row and column from the matrix to get a smaller matrix, take the determinant of that matrix, multiply by the entry in question, and multiply by ± 1 depending upon where that entry is in the matrix. Sum over all entries. In computing Laplace expansions, it is a good idea to start with a row or column which has a lot of 0's in it.

For example, consider the matrix: $\begin{bmatrix} 2 & 3 & 1 & 0 & -2 \\ 1 & 1 & 0 & 0 & 3 \\ 5 & 0 & 0 & 0 & 0 \\ 1 & 2 & -3 & 1/5 & 3 \\ 0 & 4 & 2 & 0 & 2 \end{bmatrix}$

Expanding using the 5^{th} row, you get: $5 \begin{vmatrix} 3 & 1 & 0 & -2 \\ 1 & 0 & 0 & 3 \\ 2 & -3 & 1/5 & 3 \\ 4 & 2 & 0 & 2 \end{vmatrix} + 0 + 0 + 0 + 0$

Then, expanding using the 3^{rd} column, you get: $5(0 + 0 + 1/5 \begin{vmatrix} 3 & 1 & -2 \\ 1 & 0 & 3 \\ 4 & 2 & 2 \end{vmatrix} + 0)$

Then, expanding using the 2^{nd} column, you get: $-1 \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} + 0 - 2 \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}$

Which yields: $-1(1 \times 2 - 4 \times 3) - 2(3 \times 3 - 1 \times -2) = -12$

Properties of the Determinant:

- $det(A) = det(A^t)$
- $det(AB) = det(A)det(B)$
- $det(A) = 0$ if and only if A is not invertible.
- $det(A^{-1}) = 1/det(A)$

We call the number obtained by removing the i^{th} row and the j^{th} column from a matrix A and then computing the determinant the ij-cofactor of A (denoted A_{ij}). The matrix obtained by placing A_{ji} in the ij-entry is called the adjoint of A and is denoted $adj(A)$. If we are given a matrix A, we have a formula for A^{-1}: $A^{-1} = \frac{adj(A)}{det(A)}$.

Cramer's Rule allows us to solve a system of linear equations using determinants. If we have a linear system $Ax = b$, for $x = (x_1, x_2, \cdots, x_n)$, then $x_i = \frac{det(A_i)}{det(A)}$, where A_i is the matrix obtained by replacing the i^{th} column of A with the vector b.

Practice Problems

1. Compute the following determinants:
$$\begin{vmatrix} 3 & 4 & 5 \\ 0 & 1 & -3 \\ 2 & -1 & -2 \end{vmatrix} \begin{vmatrix} 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -2 \\ 4 & 2 & 2 & 0 \\ -2 & -3 & 1 & 0 \end{vmatrix} \begin{vmatrix} -1 & 1 & 4 \\ 1 & 0 & 3 \\ -2 & 2 & 0 \end{vmatrix} \begin{vmatrix} a-b & 3a & 2a-b \\ c & d+3c & 2c \\ -f & e & -f \end{vmatrix}$$

2. Compute the following determinants:
$$\begin{vmatrix} 3 & 7 & 22 \\ 0 & 1 & -6 \\ 0 & 0 & -2 \end{vmatrix} \begin{vmatrix} 3 & 7 & 22 & 3 \\ 0 & 1 & -6 & 52 \\ 0 & 0 & -2 & -20 \\ 0 & 0 & 0 & 4 \end{vmatrix} \begin{vmatrix} 3 & 7 & 22 & 3 & -4 \\ 0 & 1 & -6 & 52 & 6 \\ 0 & 0 & -2 & -20 & -11 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix}$$
Do you notice a pattern? Can you explain why this pattern holds?

3. Compute the following determinants:
$$\begin{vmatrix} 1 & -2 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 3 & 4 \end{vmatrix} \begin{vmatrix} 1 & -2 & 13 & 11 \\ 3 & 1 & 33 & -20 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 3 & 4 \end{vmatrix} \begin{vmatrix} 1 & -2 & -1 & 5 & -2 \\ 3 & 1 & -7 & 15 & 3 \\ 0 & 0 & -2 & -1 & 4 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 1 & -2 & 2 \end{vmatrix} \begin{vmatrix} 1 & -2 & -1 & 5 & -2 \\ 3 & 1 & -7 & 15 & 3 \\ 5 & 0 & -2 & -1 & 0 \\ 3 & 0 & 3 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \end{vmatrix}$$
What is the pattern here? Why does it hold?

4. Let a, b, c, d be such that $\begin{vmatrix} a & 17 & -13 & 11 \\ b & 1 & 3 & -2 \\ c & -1 & -2 & -1 \\ d & 0 & 3 & 4 \end{vmatrix} = -3$. Calculate $\begin{vmatrix} a+2 & 17 & -13 & 11 \\ b & 1 & 3 & -2 \\ c & -1 & -2 & -1 \\ d & 0 & 3 & 4 \end{vmatrix}$.

5. If $det(A) = d$, what is the value of $det(kA)$? Can we interpret this geometrically in terms of the scaling of areas?

6. If $det(A) = d \neq 0$, calculate $det(A^{-1}A^t)$, $det((A^t)^{-1})$, and $det(adj(A))$.

7. Suppose A is a matrix with rational entries such that $A^4 = I$. What are the possible values for $det(A)$? What about if the entries of A are chosen from $\mathbb{F}_5$?

8. Suppose we have a system of n linear equations in n unknowns such that the coefficient of x_1 in each equation is equal to the constant term in that equation. What can we say about the solution to such a system of equations?

9. Say that A is a 2×3 matrix, and B is a 3×2 matrix. Do we know anything about $det(BA)$? What about $det(AB)$?

Solutions

1. $\begin{vmatrix} 3 & 4 & 5 \\ 0 & 1 & -3 \\ 2 & -1 & -2 \end{vmatrix} = -49$, $\begin{vmatrix} 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -2 \\ 4 & 2 & 2 & 0 \\ -2 & -3 & 1 & 0 \end{vmatrix} = 8$

   $\begin{vmatrix} -1 & 1 & 4 \\ 1 & 0 & 3 \\ -2 & 2 & 0 \end{vmatrix} = 8$, $\begin{vmatrix} a-b & 3a & 2a-b \\ c & d+3c & 2c \\ -f & e & -f \end{vmatrix} = adf + bce$

2. $\begin{vmatrix} 3 & 7 & 22 \\ 0 & 1 & -6 \\ 0 & 0 & -2 \end{vmatrix} = -6$, $\begin{vmatrix} 3 & 7 & 22 & 3 \\ 0 & 1 & -6 & 52 \\ 0 & 0 & -2 & -20 \\ 0 & 0 & 0 & 4 \end{vmatrix} = -24$, $\begin{vmatrix} 3 & 7 & 22 & 3 & -4 \\ 0 & 1 & -6 & 52 & 6 \\ 0 & 0 & -2 & -20 & -11 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} = 24$

   The pattern is that the determinant of an upper triangular matrix is the product of the diagonal entries.

3. Compute the following determinants:

   $\begin{vmatrix} 1 & -2 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 3 & 4 \end{vmatrix} = -35$, $\begin{vmatrix} 1 & -2 & 13 & 11 \\ 3 & 1 & 33 & -20 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 3 & 4 \end{vmatrix} = -35$

   $\begin{vmatrix} 1 & -2 & -1 & 5 & -2 \\ 3 & 1 & -7 & 15 & 3 \\ 0 & 0 & -2 & -1 & 4 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 1 & -2 & 2 \end{vmatrix} = -350$, $\begin{vmatrix} 1 & -2 & -1 & 5 & -2 \\ 3 & 1 & -7 & 15 & 3 \\ 5 & 0 & -2 & -1 & 0 \\ 3 & 0 & 3 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \end{vmatrix} = 24$

   The pattern is that if we can decompose a matrix into smaller matrices $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$, then the determinant is $det(A)det(D)$.

4. The determinant of the new matrix is $-3 + 2(13) = 23$.

5. If A is an $n \times n$ matrix, $det(kA) = k^n d$. Geometrically, this means that if we scale a shape by a factor of k in every direction, its volume increases by k^n.

6. $det(A^{-1}A^t) = 1$, $det((A^t)^{-1}) = 1/d$, and $det(adj(A)) = d^{n-1}$.

7. If A is a matrix with rational entries, $det(A)$ could be 1 or -1. If the entries of A are chosen from $\mathbb{F}_5$, $det(A)$ could be 1, 2, 3, or 4.

8. The solution to such a system of linear equations is $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, etc.

9. $det(BA)$ must be 0. $det(AB)$ could be any number.