Material Covered

A basis in $\mathbb{R}^n$ is a list of n vectors. For example, we can denote by $\mathfrak{A}$ the standard basis $[e_1, e_2, e_3]$ of $\mathbb{R}^3$, or we can denote by $\mathfrak{B}$ another basis such as $[b_1, b_2, b_3]$, where, in the standard basis, $b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $b_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Any vector which can be expressed in one basis can also be expressed in another. For example, consider the vector v which is $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ in the standard basis $\mathfrak{A}$. Since $v = 3/2b_1 + b_2 - 1/2b_3$, $v = \begin{bmatrix} 3/2 \\ 1 \\ -1/2 \end{bmatrix}$ in the basis $\mathfrak{B}$. The statement "$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in basis $\mathfrak{B}$" we abbreviate by $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{\mathfrak{B}}$.

If we have a vector expressed in a new basis $\mathfrak{B} = [b_1, b_2, \cdots, b_n]$, and we wish to express it in an old basis $\mathfrak{A} = [a_1, a_2, \cdots, a_n]$, we use a change of basis matrix. To calculate the change of basis matrix, we express each of the new basis vectors in terms of the old ones: $b_i = m_i^1 a_1 + m_i^2 a_2 + \cdots + m_i^n a_n$. The resulting matrix

$$M = \begin{bmatrix} m_1^1 & m_2^1 & \cdots & m_n^1 \\ m_1^2 & m_2^2 & \cdots & m_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ m_1^n & m_2^n & \cdots & m_n^n \end{bmatrix}$$

is the change of basis matrix, and if we have an expression $v_{\mathfrak{B}}$ for v in basis $\mathfrak{B}$, then we can find an expression for v in $\mathfrak{A}$ by $v_{\mathfrak{A}} = Mv_{\mathfrak{B}}$. We can also denote M by $\mathfrak{B}_{\mathfrak{A}}$ (the basis $\mathfrak{B}$ expressed in terms of $\mathfrak{A}$), in which case we have $v_{\mathfrak{A}} = \mathfrak{B}_{\mathfrak{A}} v_{\mathfrak{B}}$.

If we want to change back from the old basis to the new basis, we use the matrix $\mathfrak{A}_{\mathfrak{B}}$ obtained by expressing $\mathfrak{A}$ in terms of $\mathfrak{B}$. It can easily be verified that this matrix is just M^{-1}.

The change of basis matrix can be also be used to calculate the matrix of a linear map under different bases. For example, say we have a linear map T from $\mathbb{R}^n$ to $\mathbb{R}^m$, which has matrix $T_{\mathfrak{B}}^{\mathfrak{A}}$ (also denoted ${}_{\mathfrak{B}}T_{\mathfrak{A}}$) under basis $\mathfrak{A}$ of $\mathbb{R}^n$ and $\mathfrak{B}$ of $\mathbb{R}^m$. Then if we have new bases C of $\mathbb{R}^n$ and D of $\mathbb{R}^m$, $T_{\mathfrak{D}}^{\mathfrak{C}} = \mathfrak{B}_{\mathfrak{D}} T_{\mathfrak{B}}^{\mathfrak{A}} \mathfrak{C}_{\mathfrak{A}}$, where $\mathfrak{B}_{\mathfrak{D}}$ and $\mathfrak{C}_{\mathfrak{A}}$ are the appropriate change of basis matrices. Where T is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $M = \mathfrak{B}_{\mathfrak{A}}$, we get that $T_{\mathfrak{B}}^{\mathfrak{B}} = M^{-1} T_{\mathfrak{A}}^{\mathfrak{A}} M$. For matrices M and A, we call the operation of taking $M^{-1}AM$ the conjugation of A by M.

Practice Problems

1. Express each of the following vectors (given in the standard basis) in terms of the basis $\mathfrak{B} = \left[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right]$: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. Calculate the change of basis matrix $\mathfrak{B}_{\mathfrak{A}}$ in each case (every column vector is expressed in terms of the standard basis):
$\mathfrak{A} = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$ $\mathfrak{B} = \left[\begin{bmatrix} 1/3 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \end{bmatrix} \right]$, $\mathfrak{A} = \left[\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right]$ $\mathfrak{B} = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$,
$\mathfrak{A} = \left[\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right]$ $\mathfrak{B} = \left[\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right]$

3. Let $\mathfrak{A}$ be a basis $[a_1, a_2, a_3]$ of $\mathbb{R}^3$. Let $b_1 = a_1 - a_3$, $b_2 = a_1 - a_2$, and $b_3 = a_1 + a_2 + a_3$, so that $\mathfrak{B} = [b_1, b_2, b_3]$ is also a basis of $\mathbb{R}^3$. Let T be a linear map whose matrix in basis $\mathfrak{A}$ is $\begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & -3 \\ 2 & -1 & -2 \end{bmatrix}$ Calculate the matrix of T in basis $\mathfrak{B}$.

4. We wish to find a formula for the projection of a vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in R^2 onto the line spanned by $\begin{bmatrix} 1 \\ m \end{bmatrix}$ for any m.
a) Find a basis $\mathfrak{B}$ for which it will be easy to describe the projection in question (HINT: try a pair of perpendicular vectors).
b) Let T be the linear map corresponding to this projection. Find the matrix of T in basis $\mathfrak{B}$.
c) Find the matrix of T in the standard basis.

5. Using an approach similar to question 4, find the matrix of the linear map corresponding to rotation by an angle of θ about the line spanned by $e_1 + e_2 + e_3$. (HINT: The matrix corresponding to the rotation by an angle of θ about the line spanned by e_3 is $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.)

6. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, and $N = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then, let:
$B = N^{-1}AN = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 \\ 5/2 & 3/2 \end{bmatrix}$.
Calculate B^2, B^3, and B^n.

Solutions

1. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathfrak{B}}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}_{\mathfrak{B}}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{\mathfrak{B}}, \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{\mathfrak{B}},$
$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ -1/2 \end{bmatrix}_{\mathfrak{B}}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1/2 \\ -1 \end{bmatrix}_{\mathfrak{B}}$

2. $\begin{bmatrix} 1/3 & 7 \\ 4 & 6 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/5 & 1/5 \\ -2/5 & 3/5 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/4 & 1 \\ 1/2 & 1/4 & 2 \\ 1/2 & 1/4 & -2 \end{bmatrix}$

3. $T_{\mathfrak{B}}^{\mathfrak{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & -3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7/3 & -8/3 & 4 \\ -4/3 & 4/3 & 5 \\ 5/3 & 1/3 & 3 \end{bmatrix}$

4. a) $\mathfrak{B} = \left[\begin{bmatrix} 1 \\ m \end{bmatrix}, \begin{bmatrix} -m \\ 1 \end{bmatrix} \right]$

   b) $T_{\mathfrak{B}}^{\mathfrak{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} 0$

   c) $T = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{bmatrix}$

5. Use the basis $\mathfrak{B} = \left[\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ -\sqrt{2} \end{bmatrix}, \begin{bmatrix} \sqrt{3/2} \\ -\sqrt{3/2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]$.

   Multiplying out $\begin{bmatrix} 1/\sqrt{2} & \sqrt{3/2} & 1 \\ 1/\sqrt{2} & -\sqrt{3/2} & 1 \\ -\sqrt{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & \sqrt{3/2} & 1 \\ 1/\sqrt{2} & -\sqrt{3/2} & 1 \\ -\sqrt{2} & 0 & 1 \end{bmatrix}^{-1}$

   will yield the answer.

6. $B^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5/2 & 3/2 \\ 3/2 & 5/2 \end{bmatrix},$

   $B^3 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7/2 & 9/2 \\ 9/2 & 7/2 \end{bmatrix},$

   $B^n = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{2^n}{2} + \frac{(-1)^n}{2} & \frac{2^n}{2} - \frac{(-1)^n}{2} \\ \frac{2^n}{2} - \frac{(-1)^n}{2} & \frac{2^n}{2} + \frac{(-1)^n}{2} \end{bmatrix}$