Material Covered

Given a linear map T from $\mathbb{R}^n$ to $\mathbb{R}^n$, an eigenvector of T is a vector v such that Tv is a scalar multiple of v. If $Tv = \lambda v$ then v is called an eigenvector of T of eigenvalue λ. For example, if T is the identity map, every vector is a eigenvector of eigenvalue 1. If T is the map from $\mathbb{R}^3$ to $\mathbb{R}^3$ which multiplies by 3 the z-coordinate of any point, anything in the xy-plane is an eigenvector of eigenvalue 1, and e_3 is an eigenvector of eigenvalue 3. If T is the map from $\mathbb{R}^3$ to $\mathbb{R}^3$ which rotates everything by 90° about the z-axis, then the only eigenvectors are multiples of e_3 (which all have eigenvalue 1).

To find the eigenvalues of a linear map, one must calculate the characteristic polynomial. If we expand the formula $det(T - \lambda I) = 0$ in terms of λ we will get a polynomial, which can then be factored. The roots of this polynomial are the eigenvalues of T. To calculate the eigenvectors for a given λ, we need to find the solution vectors v to the system of linear equations $(\lambda I - T)v = 0$.

For example, let's start with the matrix $A = \begin{bmatrix} -1/4 & -5/4 & 2 \\ 3/4 & 7/4 & -2 \\ 3/4 & -1/4 & 0 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -1/4-\lambda & -5/4 & 2 \\ 3/4 & 7/4-\lambda & -2 \\ 3/4 & -1/4 & -\lambda \end{bmatrix}$$

$$\begin{aligned} det(A - \lambda I) &= (-1/4-\lambda)(7/4-\lambda)(-\lambda) + 15/8 - 3/8 - 1/2(-1/4-\lambda) - 3/2(7/4-\lambda) - 15/16\lambda \\ &= -\lambda^3 + 3/2\lambda^2 + 3/2\lambda - 1 \\ &= -(\lambda+1)(\lambda-2)(\lambda-1/2) \end{aligned}$$

So, A has eigenvalues -1, 2, and 1/2. To find an eigenvector with eigenvalue -1, we need to solve the system $(A+I)v = 0$. $A + I = \begin{bmatrix} 3/4 & -5/4 & 2 \\ 3/4 & 11/4 & -2 \\ 3/4 & -1/4 & 1 \end{bmatrix}$, which has rref $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

We find that the vector $v = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ satisfies the equations in question and hence is an eigenvector of eigenvalue -1.

Eigenvectors and eigenvalues can be useful in cases in which a linear map is applied repeatedly to a vector. If we wish to evaluate $T^n v$ for some large v, we first decompose v as a linear combination $v = a_1 v_{\lambda_1} + a_2 v_{\lambda_2} + \cdots$ such that each v_{λ_i} is an eigenvector of eigenvalue λ_i. Then, $T^n v = a_1 T^n v_{\lambda_1} + a_2 T^n v_{\lambda_2} + \cdots = a_1 \lambda_1^n v_{\lambda_1} + a_2 \lambda_2^n v_{\lambda_2} + \cdots$. This gives us a way to evaluate the iterated application of a linear map.

Practice Problems

1. For each linear map (described geometrically), find at least one eigenvector and its eigenvalue:
 a) the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which scales everything by a factor of k
 b) the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which reflects the plane in the line $y = 2x$
 c) the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which projects everything onto the x-axis, and then rotates the plane counterclockwise by 90°.
 d) the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which sends e_1 to $-e_1 + e_2$ and e_2 to $-e_2$
 e) the map from $\mathbb{R}^3$ to $\mathbb{R}^3$ which reflects every vector in the origin
 f) the map from $\mathbb{R}^3$ to $\mathbb{R}^3$ which sends e_1 to $2e_1 + e_2 + e_3$, e_2 to $e_1 + 2e_2 + e_3$, and e_3 to $e_1 + e_2 + 2e_3$

2. Find the eigenvalues of each matrix:
 a) $\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$; b) $\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$; c) $\begin{bmatrix} 0 & 4 \\ 2 & -3 \end{bmatrix}$; d) $\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$; e) $\begin{bmatrix} 3 & 1 & -2 \\ 0 & 1 & 4 \\ 0 & 3 & 0 \end{bmatrix}$; f) $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & -4 \\ -1 & 3 & -2 \end{bmatrix}$

3. For each A and λ, find an eigenvector of A of eigenvalue λ:
 a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\lambda = 3$; b) $A = \begin{bmatrix} 4 & 4 \\ -1 & 0 \end{bmatrix}$, $\lambda = 2$;
 c) $A = \begin{bmatrix} -1/4 & -5/4 & 2 \\ 3/4 & 7/4 & -2 \\ 3/4 & -1/4 & 0 \end{bmatrix}$, $\lambda = 2$

4. If the determinant of a matrix is zero, what can we say about its eigenvalues?

5. If a matrix is diagonal, what can we say about its eigenvalues?

6. Say that we know that $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}^{-1}$. What are the eigenvalues and eigenvectors of A?

7. Suppose that some matrix A has an eigenvector v of eigenvalue λ. Find an eigenvector and the corresponding eigenvalue of A^2 and A^3.

8. Suppose that a certain species of rabbit lives for three years, and produces exactly three offspring when it is two years old. In other words, in a year's time, a one-year-old rabbit will become a two-year-old rabbit, a two-year-old rabbit will become a three-year-old rabbit and produce three one-year-old offspring, and a three-year-old rabbit will die.
 a) Find a ratio between the ages of rabbits that is unaffected by the passage of time.
 b) What will be the result, after 10 years, of an initial population of 10 two-year-old and 5 one-year-old rabbits?

Solutions

1. a) every vector in $\mathbb{R}^2$ is an eigenvector of eigenvalue k
   b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of eigenvalue 1, and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is has eigenvalue -1
   c) anything on the y-axis is an eigenvector of eigenvalue 0
   d) e_2 is an eigenvector of eigenvalue -1
   e) everything in $\mathbb{R}^3$ is an eigenvector of eigenvalue -1
   f) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of eigenvalue 4

2. Find the eigenvalues of each matrix:
   a) $2, 2$; b) $0, -1$; c) $-\frac{3}{2} \pm \frac{\sqrt{41}}{2}$; d) 2; e) $3, -3, 4$; f) cannot be solved by hand

3. For each A and λ, find an eigenvector of A of eigenvalue λ:
   a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; b) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$; c) $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

4. One of the eigenvalues will have to be 0.

5. The eigenvalues will be the diagonal entries of the matrix.

6. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ will be an eigenvector of eigenvalue -1, and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ will have eigenvalue 2.

7. A^2 will have v as an eigenvector of eigenvalue λ^2 and A^3 will have v as an eigenvector of eigenvalue λ^3.

8. Let $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
   a) $\begin{bmatrix} 2 \\ \sqrt{2} \\ 1 \end{bmatrix}$ is an eigenvector of A of eigenvalue $\sqrt{2}$, so the ratio of 2 one-year-olds for every $\sqrt{2}$ two-year-olds to every 1 three-year-old will be unaffected over time. b)
   $$A^{10} \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix} = A^{10}\left(\left(\tfrac{5}{\sqrt{2}} + \tfrac{5}{4}\right) \begin{bmatrix} 2 \\ \sqrt{2} \\ 1 \end{bmatrix} + \left(-\tfrac{5}{\sqrt{2}} + \tfrac{5}{4}\right) \begin{bmatrix} 2 \\ -\sqrt{2} \\ 1 \end{bmatrix} - \tfrac{5}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) =$$
   $$= \left(\tfrac{5}{\sqrt{2}} + \tfrac{5}{4}\right)\sqrt{2}^{10} \begin{bmatrix} 2 \\ \sqrt{2} \\ 1 \end{bmatrix} + \left(-\tfrac{5}{\sqrt{2}} + \tfrac{5}{4}\right)(-\sqrt{2})^{10} \begin{bmatrix} 2 \\ -\sqrt{2} \\ 1 \end{bmatrix} - 0 = \begin{bmatrix} 160 \\ 320 \\ 80 \end{bmatrix}.$$
   After 10 years, there will be 160 one-year-olds, 320 two-year-olds, and 80 three-year-olds.