Material Covered

For a matrix A, we define the eigenspace of eignevalue λ to be the set of all eigenvectors of A of eigenvalue λ (together with the zero vector). The eigenspace of eigenvalue λ is the kernel of the matrix $A - \lambda I$, and is a subspace. To compute the eigenspace for a given eigenvalue, we solve the system of linear equations given by $(A - \lambda I)v = 0$. The solution space to this system is the eigenspace of λ.

Eigenvalues have both algebraic and geometric multiplicities. The algebraic multiplicity of an eigenvalue λ_0 is the number k such that $(\lambda - \lambda_0)^k$ (but not $(\lambda - \lambda_0)^{k+1}$) is a factor of the characteristic polynomial of A. The geometric multiplicity of λ is the dimension of the eigenspace of eigenvalue λ. For any eigenvalue, the geometric multiplicity is always less than or equal to the algebraic multiplicity.

A matrix A is said to be diagonalizable if there is an invertible matrix N and a diagonal matrix D such that $A = NDN^{-1}$ (this is the same as saying $D = N^{-1}AN$). Such a matrix D will always have the eigenvalues of A on the diagonal, and the columns of N will be a basis consisting of eigenvectors of A. We will always be be able to find such a matrix N if the algebraic and geometric multiplicities of all the eigenvalues are equal, as we will be able to make a basis of the entire space out of bases of each of the eigenspaces. However, if there is some eigenvalue for which the geometric multiplicity is less than the algebraic multiplicity, the matrix is not diagonalizable.

For example, say we wish to diagonalize the matrix $A = \begin{bmatrix} -4 & 6 & 6 \\ 0 & 2 & 0 \\ -3 & 3 & 5 \end{bmatrix}$. We first calculate the characteristic polynomial $det(\lambda I - A) = (\lambda - 2)^2(\lambda + 1)$. So, we know we have one eigenvalue, 2, of algebraic multiplicity 2, and another, -1 of algebraic mutliplicity 1. We then have to find the eigenspaces for each eigenvalue. To find the eigenspace for 2 we solve the linear system $(A - 2I)v = 0$. We find that the eigenspace for 2 consists of all vectors satisfying $v_1 = v_2 + v_3$. To find the eigenspace for -1 we solve the linear system $(A + I)v = 0$. The eigenspace for -1 consists of vectors satisfying $v_1 = 2v_3$ and $v_2 = 0$.
$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ have eigenvalue 2 and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ has eigenvalue 1. Thus $\left[\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right]$ is a basis of eigenvectors, and $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$.

We call $n \times n$ matrices A and B similar if there is an invertible matrix N such that $A = NBN^{-1}$. A and B will have the same eigenvalues, characteristic polynomial, trace (sum of the diagonal entries) and determinant. In the characteristic polynomial $det(\lambda I - A)$, the coefficient of λ^{n-1} is $-Tr(A)$ and the constant term is $(-1)^n det(A)$.

Practice Problems

1. For each of the matrices below, find the eigenvalues, and the algebraic multiplicity of each eigenvalue:
a) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 1 & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$, c) $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, d) $\begin{bmatrix} -4 & -4 & 2 \\ 3 & 4 & -1 \\ -3 & -2 & 3 \end{bmatrix}$, e) $\begin{bmatrix} -2 & 1 & -2 \\ 5 & -3 & 6 \\ 5 & -1 & 4 \end{bmatrix}$

2. For each matrix from question 1 find the geometric multiplicities of each of the eigenvalues.

3. Which matrices from question 1 are diagonalizable? Express each of the diagonalizable matrices as a product NDN^{-1} for D diagonal and N invertible.

4. Let the linear map T be a reflection about a hyperplane of dimension $n-1$ in R^n. Explain why T must be diagonalizable. (Hint: Start with the case $n = 2$).

5. Let the matrix A be such that $A-2I$ has a 1-dimensional kernel and A has characteristic polynomial $\lambda^3 - \lambda^2 - 8\lambda + 12$. Is A diagonalizable?

6. For which t is the matrix $\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ diagonalizable?

7. For which p, q, and r is the matrix $\begin{bmatrix} 2 & p & q \\ 0 & 2 & r \\ 0 & 0 & 2 \end{bmatrix}$ diagonalizable?

8. A 3×3 matrix has a 2-dimensional kernel, and has trace equal to 1. Is such a matrix necessarily diagonalizable? What are its eigenvalues?

9. Explain why it is impossible for an eigenvalue to have geometric multiplicity 0.

10. List all the matrices similar to $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$.

11. Let A and B be 4×4 matrices, each with the eigenvalues 1, 3, -2 and 1. Explain why A and B must be similar.

12. Find an example of two matrices which both have the same characteristic polynomial, but are not similar.

Solutions

1. a) -1 of algebraic multiplicity 1, 3 of a.m. 1; b) $\frac{1}{3}$ of a.m. 1, $\frac{1}{2}$ of a.m. 1; c) 1 of a.m. 1, 3 of a.m. 1, 4 of a.m. 1; d) -1 of a.m. 1, 2 of a.m. 2; e) -2 of a.m. 1, $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ each of a.m. 1

2. Every eigenvalue for a), b), c), and e) has algebraic multiplicity 1, so it must have geometric multiplicity 1. For d), a kernel calculation shows that -1 has g.m. 1, and 2 has g.m. 2.

3. Every matrix is diagonalizable. Diagonalizations:
   a) $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$; b) $\begin{bmatrix} 1 & \frac{4}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & \frac{4}{3} \\ 1 & 1 \end{bmatrix}^{-1}$;
   c) $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$;d) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix}^{-1}$;
   e) $\begin{bmatrix} 1 & 1 & 1 \\ -\frac{5}{2} & -\frac{5}{2} - \frac{\sqrt{5}}{2} & -\frac{5}{2} + \frac{\sqrt{5}}{2} \\ -\frac{5}{4} & -\frac{5}{2} - \frac{\sqrt{5}}{2} & -\frac{5}{2} + \frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & \frac{1}{2} + \frac{\sqrt{5}}{2} & 0 \\ 0 & 0 & \frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -\frac{5}{2} & -\frac{5}{2} - \frac{\sqrt{5}}{2} & -\frac{5}{2} + \frac{\sqrt{5}}{2} \\ -\frac{5}{4} & -\frac{5}{2} - \frac{\sqrt{5}}{2} & -\frac{5}{2} + \frac{\sqrt{5}}{2} \end{bmatrix}^{-1}$

4. T must be diagonalizable because its geometric multiplicities will sum to n (and thus must equal the algebraic multiplicities. The $(n-1)$-dimensional hyperplane of reflection consists of eigenvectors of eigenvalue 1, so 1 has geometric multiplicity $n-1$. Also, the line perpendicular to the plane consists of eigenvectors of eigenvalue -1, so -1 has geometric multiplicity 1. Thus, we can find a basis of eigenvectors and T is diagonalizable.

5. A is not diagonalizable because the eigenvalue 2 has geometric multiplicity 1 and algebraic multiplicity 2.

6. t must be 0.

7. p, q, and r must all be 0.

8. Such a matrix must necessarily be diagonalizable and must have 0 (of algebraic and geometric multiplicity 2) and 1 (of algebraic and geometric multiplicity 1) as eigenvalues.

9. If an eigenvalue had geometric multiplicity 0, there would be no eigenvectors corresponding to that eigenvalue, and thus it would not be an eigenvalue.

10. $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ is the only matrix similar to itself.

11. A and B must both be diagonalizable because they both have 4 eigenvalues each of algebraic multiplicity 1. Since they both have the same eigenvalues, they are both similar to the same diagonal matrix, so they are similar to each other.

12. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have the same characteristic polynomial, but are not similar because one is diagonalizable, and the other is not.