Material Covered

Sometimes a characteristic polynomial of degree n will not completely factor to yield n real eigenvalues. In this case, we must introduce the complex numbers in order to completely factor the polynomial and find n eigenvalues.

Complex numbers are numbers of the form $a + bi$ where a and b are real numbers and i is defined to be the square root of -1. Complex numbers can also be expressed in the form $r(\cos\theta + i\sin\theta)$ where r is the "length" of the complex number, and θ is the "angle". To add and multiply complex numbers, you use addition and multiplication of real numbers (plus the fact that $i^2 = -1$). i.e. $(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$ and $(a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i$. Multiplication and taking powers of complex numbers is easier when the numbers are expressed in polar form $r(\cos\theta + i\sin\theta)$. $r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2) = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$ - when you multiply complex numbers you multiply the lengths and add the angles. In taking powers, you get: $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$.

Any polynomial of degree n with real or complex coefficients will factor completely in the complex numbers to yield n roots. This means that any $n \times n$ matrix will have complex eigenvalues of algebraic multiplicities summing to n. So, if we work over the complex numbers, the only obstacle to the diagonalization of a matrix will be if the geometric and algebraic multiplicities are not equal.

Linear algebra can be done using the complex numbers, just as it can using the real numbers or using a finite field. Matrices with complex entries have eigenvalues and eigenvectors, just like real matrices. They can be reduced to reduced-row-echelon form, they have kernels, determinants and all other properties that matrices have.

For example, consider the matrix $A = \begin{bmatrix} 1+i & 1 \\ 2i & 1+i \end{bmatrix}$. A has characteristic polynomial $\lambda^2 - (2+2i)\lambda$, so it has eigenvalues 0 and $2+2i$. To find the eigenvector for eigenvalue 0 we must row-reduce $A - 0I = A$, which has rref $\begin{bmatrix} 1 & \frac{1}{2} - \frac{1}{2}i \\ 0 & 0 \end{bmatrix}$, and thus $\begin{bmatrix} -\frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix}$ is an eigenvector of eigenvalue 0. To find an eigenvector of eigenvalue $2+2i$, we row-reduce $\begin{bmatrix} -1-i & 1 \\ 2i & -1-i \end{bmatrix}$ to get rref $\begin{bmatrix} 1 & -\frac{1}{2} + \frac{1}{2}i \\ 0 & 0 \end{bmatrix}$, and thus $\begin{bmatrix} \frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix}$ is an eigenvector of eigenvalue $2 + 2i$.

We call the eigenvalue of a matrix which has the largest length the *dominant eigenvalue*. We can compute the dominant eigenvalue by iterating the matrix. If we pick some initial vector v_0 (this method will work for most, but not all initial vectors v_0), and apply the matrix A to v_0 n times, the vector $A^n\, v_0$ will be close to a dominant eigenvector of A, and will get closer to such an eigenvalue as n grows. The dominant eigenvalue will be approximated by the length of $A^{n+1}v_0$ divided by the length of $A^n v_0$.

Practice Problems

1. Express the following complex numbers in the form $a+bi$:
   a) $(2+3i)+(-3-2i)$; b) $(2+3i)(-3-2i)$; c)$\frac{1}{2+i}$; d) $\frac{1+i}{1-i}+\frac{1-i}{1+i}$; e) $(\sqrt{3}-i)^5$

2. Draw, as points on the plane, the numbers z such that $z^7=1$.

3. Find the eigenvalues and eigenvectors of the following matrices. Which are diagonalizable? a) $\begin{bmatrix} 1+i & 2 \\ 0 & 1+i \end{bmatrix}$; b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ c) $2\begin{bmatrix} -3 & 4 \\ -2 & 1 \end{bmatrix}$; d) $\begin{bmatrix} -1+i & 1 \\ 1+i & -1 \end{bmatrix}$;
   e) $\begin{bmatrix} 1 & -3 & 11 \\ 2 & -6 & 16 \\ 1 & -3 & 7 \end{bmatrix}$; f) $\begin{bmatrix} 0 & 0 & 1+i \\ -1-i & 0 & 0 \\ 0 & \frac{i}{2} & 0 \end{bmatrix}$

4. Calculate the determinant of the matrix:
   $$\begin{bmatrix} \frac{2}{3} & 4i & 1+i & 1+5i & -\frac{i}{6} \\ i & -i & -2 & 3i & -1-\frac{i}{2} \\ 3 & 3 & 12+6i & -4i & \frac{3i}{4} \\ 1 & -1 & 2i & 3 & -\frac{1}{2}+i \\ 1+3i & \sqrt{5}i & -41 & 6+i & 4-i \end{bmatrix}$$

5. If a matrix A is such that $A^4=I$, what are the possible eigenvalues for A?

6. Say we have a 4×4 matrix A such that A^n approaches $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ as n approaches infinity. What does this tell us about the eigenvalues of A? What does it tell us about the eigenvectors?

7. Say we have a 2×2 matrix A such that, $A^{50}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 10013 \\ 19843 \end{bmatrix}$, $A^{51}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 20018 \\ 39796 \end{bmatrix}$, $A^{-50}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} -24653 \\ 10234 \end{bmatrix}$, and $A^{-51}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 74361 \\ -30305 \end{bmatrix}$. What do we think are the eigenvectors and eigenvalues of A? What would be the matrix A in this case?

8. Find an example of a3 $\times$ 3 matrix with exactly one real eigenvalue which is not diagonalizable.

9. The matrix which rotates the plane $\mathbb{R}^2$ by an angle θ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Find the eigenvalues of this matrix for $\theta=\pi$, $\theta=\frac{\pi}{2}$, $\theta=\frac{\pi}{3}$ and $\theta=\frac{\pi}{4}$. What is the relationship between the eigenvalues and the angle θ?

Solutions

1. a) $-1+i$; b) $-13i$; c)$\frac{2}{5}-\frac{i}{5}$; d) 0; e) $-16\sqrt{3}-16i$

2. There are 7 such points evenly spaced around the unit circle at angles corresponding to $\frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, 2\pi$.

3. a) eigenvalue $1+i$, corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, not diagonalizable;

   b) $\{i,-i\}$, $\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$; c) $\{1+2i, 1-2i\}$, $\left\{\begin{bmatrix} 1-i \\ 1 \end{bmatrix}, \begin{bmatrix} 1+i \\ 1 \end{bmatrix}\right\}$, ;

   d) $\{-2i,-i\}$, $\left\{\begin{bmatrix} 1 \\ -1-i \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$; e) $\{1+i, 1-i, 0\}$, $\left\{\begin{bmatrix} 3-2i \\ 3-i \\ 1 \end{bmatrix}, \begin{bmatrix} 3+2i \\ 3+i \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}\right\}$;

   f) $\{-\frac{1}{2}+\frac{i\sqrt{3}}{2}, -\frac{1}{2}-\frac{i\sqrt{3}}{2}, 1\}$, $\left\{\begin{bmatrix} \frac{-1+\sqrt{3}}{2}-\frac{1+\sqrt{3}}{2} \\ \sqrt{3}+i \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{-1-\sqrt{3}}{2}-\frac{1-\sqrt{3}}{2} \\ -\sqrt{3}+i \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1-i \\ \frac{1}{2}-\frac{i}{2} \end{bmatrix}\right\}$;

   all matrices b) - f) are diagonalizable

4. Rows 2 and 4 are multiples of each other, so the determinant is 0.

5. The possible eigenvalues of A are the 4^{th} roots of 1: $1, -1, i, -i$.

6. We know that 1 must be an eigenvalue of A and that all other eigenvalues have length less than 1. We know that e_1 is an eigenvector of eigenvalue 1, and that the vectors e_2, e_3, e_4, when expressed as linear combinations of eigenvectors, don't involve the vector e_1.

7. It looks like 2 is the dominant eigenvalue of A, and that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the corresponding eigenvector, and that -3 is the dominant eigenvalue of A^{-1} and that $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ is the corresponding eigenvector.

   This means that $A = \begin{bmatrix} 1 & -5 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{18} & \frac{35}{36} \\ \frac{7}{9} & \frac{29}{18} \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has only one real eigenvalue (1), but is not diagonalizable.

9. The eigenvalues will be $\cos\theta \pm i\sin\theta$, which correspond to the complex numbers of length 1 and angle $\pm\theta$.