Material Covered

Three applications of Eigenvectors and Eigenvalues were covered in the last week: dominant eigenvalues, Gerschgorin disks, and stochastic matrices.

The eigenvalue of a matrix which has the largest length is called the *dominant eigenvalue*. The dominant eigenvalue governs what happens when you iterate the matrix a large number of times. If we take a vector v_0, and apply the matrix A, rescale the vector so its largest coordinate is 1, and then repeat this process over and over, we will approach an eigenvector of A corresponding to the dominant eigenvalue. This process will allow the dominant eigenvalue of any matrix to be calculated without factoring the characteristic polynomial.

The Gerschgorin Disk Theorem gives us a method of estimating the eigenvalues of a matrix based solely on the entries of the matrix. For the i^{th} row of a matrix A we form a Gerschgorin disk in the complex plane with center at a_{ii} and radius equal to $\sum_{j\neq i} \|a_{ij}\|$. The Gerschgorin Disk Theorem states that every eigenvalue of A will be contained in exactly one Gerschgorin Disk. Whenever we have k of these Gerschgorin Disks which intersect each other, but do not interesect any other disks, there will be exactly k eigenvalues of A contained in the union of these of these disks. We can also form Gerschgorin Disks using the columns, rather than the rows, of A.

For example, consider the matrix $A = \begin{bmatrix} 4 & 3 & 2 \\ 0 & -2 & \frac{1}{2} \\ 0 & \frac{1}{3} & -3 \end{bmatrix}$. Considering the Gerschgorin disks corresponding to the rows, we get one disk of radius 5 centered at 4, one of radius $\frac{1}{2}$ centered at -2, and one of radius $\frac{1}{3}$ centered at -3. Since all three disks are disjoint, all eigenvalues must be distinct and exactly one must be contained in each disk. Considering the Gerschgorin disks corresponding to the columns, we get one disk centered at 4 of radius 0, one of radius $\frac{10}{3}$ centered at -2 and one of radius $\frac{5}{2}$ centered at 3. The second two disks don't give us any extra information, but the disk at 4 of radius 0 tells us that one eigenvalue is equal to 4. We can conclude that the eigenvalues are 4, something within $\frac{1}{2}$ of -2 and something within $\frac{1}{3}$ of -3.

A Stochastic Matrix is a matrix which contains only non-negative entries between 0 and 1 (inclusive), and in which each column sums to 1. The matrix can be seen as a matrix of probabilities corresponding to a transition from an old state to a new state. The entry in the i^{th} row and j^{th} column corresponds to the probability of getting to the i^{th} new state from the j^{th} old state. By the Gerschgorin disk theorem, we can prove that all eigenvalues of a stochastic matrix will be contained in the unit circle, and, if all diagonal entries of the matrix are non-zero, then the only possible eigenvalue on the unit circle is 1. This means that 1 will be the dominant eigenvalue of a Stochastic matrix, and thus (at least most of the time), if a Stochastic matrix is iterated over and over, the resulting vector will approach a limit.

Practice Problems

1. Draw the Gerschgorin disks for the following matrices and indicate how many eigenvalues must be located in each region:

a) $\begin{bmatrix} 3 & \frac{1}{2} \\ 2 & -1 \end{bmatrix}$; b) $\begin{bmatrix} 1 & 2 & -1 \\ \frac{1}{2} & \frac{3}{2} & 0 \\ 1 & \frac{1}{3} & 3 \end{bmatrix}$; c) $\begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 3 & -\frac{5}{2} & -1 \\ 1 & -1 & 3 \end{bmatrix}$;

d) $\begin{bmatrix} i & 1+i & 1-i \\ \frac{i}{2} & -4-i & 1 \\ -i & 0 & 5 \end{bmatrix}$; e) $\begin{bmatrix} 1+2i & \frac{i}{2} & -1 & 0 \\ \frac{1}{3} & 1-i & 0 & \frac{-1}{4} \\ i & -i & 2+3i & \frac{1}{4} \\ 1 & \frac{1}{2} & 0 & -2i \end{bmatrix}$

2. Find the dominant eigenvalue and a correspoding eigenvector for each of the following matrices:

a) $\begin{bmatrix} 2 & -1 \\ \sqrt{2}-3 & \sqrt{2} \end{bmatrix}$; b) $\begin{bmatrix} -8 & 1 & 1 \\ 5 & -11 & -4 \\ 3 & -2 & -15 \end{bmatrix}$; c) $\begin{bmatrix} 206+133i & 8+94i & 52-114i \\ -156+317i & -133+81i & 198+64i \\ 81+83i & -42+94i & 77-39i \end{bmatrix}$

3. Consider the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. What happens when we apply A to a vector v_0 a large number of times? Does this contradict our statement about the dominant eigenvalue governing the long-term behaviour?

4. Explain why the eigenvalues of a matrix A^t are the same as those of A. Explain why the Gerschgorin disk theorem works for the columns as well as the rows of A.

5. Consider the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Is A a stochastic matrix? Do we reach a limit by applying A repeatedly to a vector v_0? What does this tell us about the eigenvalues of A?

6. Explain why, if A is a stochastic matrix, A^2 must also be a stochastic matrix. Find a matrix A such that A^2 is a stochastic matrix, but A is not.

7. Say that the matrix A is stochastic. Prove each of the following:

a) If $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $u = Av = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ then $v_1 + v_2 + \cdots + v_n = u_1 + u_2 + \cdots + u_n$

b) If v is an eigenvector of A for an eigenvalue other than 1 then $v_1 + v_2 + \cdots + v_n = 0$.

c) If, for each eigenvalue of A, there is an eigenvector with non-negative coordinates, then A must be I.

Solutions

1. a) Disks are centered at 3 and -1 of radii $\frac{1}{2}$ and 2, each contain one eigenvalue.
   b) Three eigenvalues are contained in the union of disks centered at 1, $\frac{3}{2}$, and 3, of radii 3, $\frac{1}{2}$ and $\frac{4}{3}$.
   c) Three eigenvalues are contained in the union of disks centered at 1, $-\frac{5}{2}$, and 3, of radii $\frac{3}{2}$, 4 and 2.
   d) Disks are centered at i, $-4-i$, and 5, of radii $2\sqrt{2}$, $\frac{3}{2}$, and 1, each containing one eigenvalue.
   e) The union of the disks centered at $1+2i$ and $2+3i$ of radii $\frac{3}{2}$ and $\frac{9}{4}$ contain two eigenvalues. The other two eigenvalues are contained in the union of the disks centered at $1-i$ and $2i$ of radii $\frac{7}{12}$ and $\frac{3}{2}$.

2. a) eigenvalue 3, eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; b) $-12-3\sqrt{3}$, $\begin{bmatrix} 1-\sqrt{3} \\ 1 \\ 2+\sqrt{3} \end{bmatrix}$; c) $75+175i$, $\begin{bmatrix} -2i \\ 4 \\ 1 \end{bmatrix}$

3. Applying this matrix repeatedly to a vector causes the vector to spiral out away from the origin. This happens because there is no dominant eigenvalue: there are two eigenvalues equal in magnitude.

4. The eigenvalue of A are the same as A^t because $det(\lambda I - A) = det((\lambda I - A)^t) = det(\lambda I^t - A^t) = det(\lambda I - A^t)$, so A and A^t have the same characteristic polynomial and hence the same eigenvalues. Since the eigenvalues of A^t are contained in the Gerschgorin disks corresponding to the rows of A^t, these same disks will correspond to the columns of A and contain the eigenvalues of A.

5. This matrix is stochastic, but we do not reach a limit applying it repeatedly because it does not have a single dominant eigenvalue - all four of its eigenvalues have magnitude 1.

6. If A is stochastic, it contains transition probabilities between two states after one time-step. So, A^2 (A iterated twice) will be stochastic because it contains the transition probabilities between states after two time-steps. $-I$ is not stochastic, but $(-I)^2 = I$ is.

7. a)$u_1+u_2+\cdots+u_n = \sum_{i,j} A^i_j v_i = v_1 \sum_j A^1_j + v_2 \sum_j A^2_j + \cdots + v_n \sum_j A^n_j = v_1+v_2+\cdots+v_n$
   b)If $u = Av$, $u = \lambda v$ (where λ is the eigenvalue), and thus $\lambda(v_1+v_2+\cdots+v_n) = u_1+u_2+\cdots+u_n = v_1+v_2+\cdots+v_n$, so $v_1+v_2+\cdots+v_n = 0$.
   c)If an eigenvector v has non-negative coordinates, it cannot have $v_1+v_2+\cdots+v_n = 0$, because this would imply that the $v = 0$. So, by b), all eigenvalues must be 1, and $A = I$.