Material Covered

An abstract vector space is a set of "vectors" which can be added together and multiplied by scalars in order to yield new vectors. The rules for addition and scalar multiplication have to satisfy a number of axioms in order for the set to count as a vector space. The axioms guarantee that the set will behave "like" $\mathbb{R}^n$ for some n. Some examples of abstract vector spaces are the $n \times m$ matrices (denoted by $M_{n\times m}(\mathbb{R})$), continuous infinitely differentiable real-valued functions on $\mathbb{R}$ (denoted by $C^\infty(\mathbb{R})$, and polynomials of degree up to n with coefficients in $\mathbb{R}$ (denoted by $P_n(\mathbb{R})$).

A subspace of a vector space is a subset of the vector space which is "closed" under addition and multiplication by scalars. This means that, for any two vectors v and w in the subspace, and for any scalar c, $v + w$ and cv are both in the subspace. Any subspace of a vector space is a vector space itself.

For any vector space V, a set of vectors $\{v_1, v_2, \cdots, v_n\}$ is said to span V if every vector w in v can be written in the form $w = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ for some scalars c_i. A set of vectors $\{v_1, v_2, \cdots, v_n\}$ is said to be linearly independent if the only scalars c_i for which $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ are the scalars $c_1 = c_2 = \cdots = c_n = 0$. Any subset B of a vector space V is said to be a basis of V if it both spans V and is linealy independent.

A linear transformation (linear map) from one vector space V to another W is a function A such that, for any v_1 and v_2 in V, and any scalar c, $A(v_1 + v_2) = A(v_1) + A(v_2)$ and $A(cv_1) = cA(v_1)$. Differentiation is a linear map from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$. Evaluation at a fixed point is a map from $C^\infty(\mathbb{R})$ to R. Evalutation at two points is a linear map from $C^\infty(\mathbb{R})$ to R^2. Multiplication by the function $\sin x$ is a linear map from $P_n(\mathbb{R})$ to $C^\infty(\mathbb{R})$.

The dimension of a vector space is defined to be the number of vectors in any basis of that vector space. If S is a linearly independent set of vectors in V, then S will never contain more than $dim(V)$ vectors. If S is a set of vectors that spans V, then S will never contain less than $dim(V)$ vectors. $\mathbb{R}^n$ is always n-dimensional; $P^n(\mathbb{R})$ is always $(n + 1)$-dimensional. Some vector spaces, such as $C^\infty(\mathbb{R})$, are infinite-dimensional.

If we have a linear map A from V to W, we define the range of A to be the set of all vectors in W which are equal to Av for some v in V. We define the kernel of A to be the set of all vectors v in V for which $Av = 0$. $range(A)$ is always a subspace of W, and $kernel(A)$ is always a subspace of V.

We call the dimension of the range of A the rank of A, and we call the dimension of the kernel of A the nullity of A. The rank-nullity theorem says that the rank of A plus the nullity of A is equal to the dimension of V (the domain of A). In the case where $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, this become the standard rank-nullity theorem for matrices.

Practice Problems

1. Which of the following are subspaces of the vector space in question? Why or why not?:
a) The set of functions f in $C^\infty(\mathbb{R})$ for which $\int_0^1 f(x)dx = 0$.
b) The set of polynomials in $P_n(\mathbb{R})$ which are divisible by $(x-2)$.
c) The set of invertible matrices in $M_{n\times n}(\mathbb{R})$.
d) The set $C^\infty(\mathbb{R})$ in the set $C^0(\mathbb{R})$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
e) The set of polynomials in $P_n(\mathbb{R})$ which have degree greater than d (for some $d < n$).

2. Which of the following maps are linear? Why or why not?:
a) The map which sends a polynomial in $P_n(\mathbb{R})$ to the sum of the coefficients of that polynomial in $\mathbb{R}$.
b) The map from $C^\infty(\mathbb{R})$ to itself which sends a function $f(x)$ to $f(sin(x))$.
c) The map from $M_{n\times n}(\mathbb{R})$ to $P_n(\mathbb{R})$ which sends a matrix to its characteristic polynomial.
d) The map from $C^\infty(\mathbb{R})$ to $\mathbb{R}$ which sends a function $f(x)$ to $e^f(e)$.
e) The map from $C^\infty(\mathbb{R})$ to W_1 which sends a function $f(x)$ to $e^f(e)$ (W_1 is the one-dimensional "weird" vector space which consists of all real numbers greater than zero and in which addition is multiplication and scalar multiplication is exponentiation).

3. Find a basis for each of the following vector spaces:
a) The subspace of $M_{n\times n}(\mathbb{R})$ which consists of all matrices with trace equal to 0.
b) The three-dimensional "weird" vector space W_3.
c) The subspace of $C^\infty(\mathbb{R})$ consisting of all functions f for which $f''(x) = 0$ for all x.

4. Let V be the set of functions in $C^\infty(\mathbb{R})$ spanned by $\{1, \sin x, \cos x, \sin^2 x, \cos^2 x, \sin^3 x, \sin^2 x \cos x, \sin x \cos^2 x, \cos^3 x\}$. We wish to calculate the dimension of V.
a) Prove that V is a subspace of $C^\infty(\mathbb{R})$.
b) Find a linear map A from $\mathbb{R}^9$ to $C^\infty(\mathbb{R})$ which has range equal to V.
c) What is the kernel of A? What is its dimension?
d) What does the rank-nullity theorem say about the dimension of V?

5. For any two vector spaces V and W, we can define their direct sum $V \bigoplus W$ to be the vector space of all ordered pairs (v, w) in which v comes from V and w from W. Addition is defined by $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$, and scalar multiplication by $c(v, w) = (cv, cw)$. For example, $\mathbb{R}^2 = \mathbb{R} \bigoplus \mathbb{R}$.
a) If $\{v_1, \cdots, v_n\}$ form a basis of V and $\{w_1, \cdots, w_m\}$ form a basis of W, find a basis of $V \bigoplus W$.
b) If V has dimension n and W has dimension m, what is the dimension of $V \bigoplus W$?
c) Find a linear map A from V to $V \bigoplus W$ which has a trivial kernel. What is the range of A?
d) Find a linear map B from $V \bigoplus W$ onto W. What is the kernel of B?
e) What can we say about the map BA from V to W?

Solutions

1. a) Is a subspace because of properties of integrals.
   b) Is a subspace because is the range of the linear map which multiplies everything by $(x-2)$.
   c) Not a subspace - does not contain 0.
   d) Is a subspace because $C^{\infty}(\mathbb{R})$ and $C^{0}(\mathbb{R})$ are both vector spaces.
   e) Not a subspace - the sum of two polynomials of the same degree can have a smaller degree.

2. a) Yes, it is linear, because when you add or scalar multiply polynomials, you add or scale their coefficients .
   b) Yes, it is linear, because $(cf + dg)(\sin x) = c(f(\sin x)) + d(g(\sin x))$.
   c) No, it is not linear - many counterexamples can be found.
   d) No, it is not linear because sums get sent to products.
   e) Yes, it is linear because sums get sent to products in $\mathbb{R}$, which are sums in W_1, same for scalar multiples.

3. a) One basis contains E_{ij} for all $i \neq j$ together with $E_{ii} - E_{(i+1)(i+1)}$ for $1 \leqslant i \leqslant n-1$.
   b) One basis contains $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.
   c) One basis contains 1 and x.

4. a) V is a subspace because it is a set of linear combinations, and the sum or scalar multiple of linear combinations is itself a linear combination.
   b) A sends e_1 to 1, e_2 to $\sin x$, e_3 to $\cos x$, e_4 to $\sin^2 x$, e_5 to $\cos^2 x$, e_6 to $\sin^3 x$, e_7 to $\sin^2 x \cos x$, e_8 to $\sin x \cos^2 x$, and e_9 to $\cos^3 x$
   c) The kernel of A is spanned by $e_4 + e_5 - e_1$, $e_6 + e_8 - e_2$, and $e_7 + e_9 - e_3$. It has dimension 3.
   d) The dimension of V is $9 - 3 = 6$.

5. a) A basis of $V \bigoplus W$ is $\{(v_1, 0), \cdots, (v_n, 0), (0, w_1), \cdots, (0, w_m)\}$.
   b) The dimension of $V \bigoplus W$ is $n + m$.
   c) A sends v_i to $(v_i, 0)$. The range of A is spanned by $\{(v_1, 0), \cdots, (v_n, 0)\}$.
   d) B sends $(v_i, 0)$ to 0, and $(0, w_j)$ to w_j.The kernel of B is spanned by $\{(v_1, 0), \cdots, (v_n, 0)\}$.
   e) BA sends everything in V to 0.