Material Covered

A linear map T from a vector space V to a vector space W is said to be injective if its kernel is $\{0\}$, it is said to be surjective if its range is all of W. If T is both injective and surjective, it is said to be an isomorphism from V onto W, and the vector spaces V and W are said to be isomorphic. Isomorphic vector spaces have the same vector space structure (what it exactly means for two vector spaces to have the same structure will be explored in more detail in the practice problems). For example, $P_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^{n-1}$: there is an isomorphism T which sends $\sum_i a_i e_i$ to $\sum_i a_i x^{i-1}$.

A differential equation is an equation which relates the derivative of a function $f'(x)$ to its value $f(x)$, and to x. The following are all differential equations: $f'(x) = x^2$, $f'(x) = f(x)$, $f'(x) - 2f(x) = \frac{1}{x}$, and $log(f'(x)) - \sin(f(x)) = \cos^2 x$. To solve a differential equation, one must find the function (or set of functions) $f(x)$ which satisfies the differential equation.

Solving a differential equation in general is not easy. However, a certain class of differential equations, linear differential equations, are much easier to solve. Just as systems of linear equations (which we solve for $x \in \mathbb{R}^n$) are of the form $Ax = b$ for a given vector b and a given linear map A, linear differential equations are of the form $T(f) = g$ for a linear "differential operator" T and a given function g. For example, the first three differential equations in the preceding paragraph are linear, and the last is not. This is because the maps $T(f) = f'$, $T(f) = f' - f$, and $T(f) = f\prime - 2f$ are linear while $T(f) = log(f'(x)) - \sin(f(x))$ is not.

In general, there will not always be a unique solution to a differential equation of the form $T(f) = g$. In particular, if f_1 is a solution, and $f_2 - f_1 \in ker(T)$, then f_2 is also a solution. Similarly, if f_1 and f_1 are both solutions, then $f_2 - f_1$ will be in the kernel. So, to find the set of all solutions to a linear differential equation, it is enough to find one solution and calculate the kernel of T.

Often times we are not asked to find all the solutions to a given differential equation, but only one solution which satisfies a given initial condition. For example, we could be asked to find the solution to $f'(x) + 5f(x) = x^3$ which satisfies $f(0) = 0$. Because the equation $g(x) = x^3$ is a polynomial of degree 3, we will guess that $f(x)$ is also such a polynomial. So, let $f(x) = ax^3 + bx^2 + cx + d$. Then $f'(x) = 3ax^2 + 2bx + c$, and $f'(x) + 5f(x) = 5ax^3 + (5b + 3a)x^2 + (5c + 2b)x + (5d + c)$. Since this equals x^3, $a = 1$, $(5b + 3a) = 0$, $(5c + 2b) = 0$, and $(5d + c) = 0$. So, $a = 1$, $b = -\frac{3}{5}$, $c = \frac{6}{25}$, and $d = -\frac{6}{125}$. So, one solution is $f(x) = x^3 - \frac{3}{5}x^2 + \frac{6}{25}x - \frac{6}{125}$. However, this solution does not satisfy $f(0) = 0$. So, we have to find the kernel of T. In this case $T(f) = f' + 5f$. So, $T(f) = 0$ exactly when $f' = -5f$, which occurs when $f(x) = ke^{-5x}$ for some constant k. So, the general solution is $f(x) = x^3 - \frac{3}{5}x^2 + \frac{6}{25}x - \frac{6}{125} + ke^{-5x}$, and we want to find that k for which $f(0) = 0$. $f(0) = 0 - \frac{3}{5}(0) + \frac{6}{25}(0) - \frac{6}{125} + ke^0 = k - \frac{6}{125}$. So, $f(0) = 0$ exactly when $k = \frac{6}{125}$, and the solution we are looking for is $f(x) = x^3 - \frac{3}{5}x^2 + \frac{6}{25}x - \frac{6}{125} + \frac{6}{125}e^{-5x}$.

Practice Problems

1. Which pairs of vector spaces are isomorphic? For the ones that are isomorphic, find an isomorphism between them.
   a) $M_{2\times 3}(\mathbb{R})$ and $M_{3\times 2}(\mathbb{R})$.
   b) The subspace of $M_{2\times 3}(\mathbb{R})$ which consists of all matrices with non-trivial kernel, and the subspace of $M_{3\times 2}(\mathbb{R})$ which consists of all matrices with non-trivial kernel.
   c) $\mathbb{R}$ and the kernel of $T(f) = 2f' + \frac{f}{3}$.
   d) $\mathbb{R}^2$ and $\mathbb{C}^2$. (Considered as vector spaces over $\mathbb{R}$).
   e) $\mathbb{R}^2$ and $\mathbb{C}$.
   f) $\mathbb{R}^3$ and W_3.

2. Which differential equations are linear? Why or why not?
   a) $f'(x) = \frac{1}{x}$; b) $3f'(x) - \frac{1}{f(x)} = \sin x$; c) $e^x - f'(x) = 5x$; d) $f'(x) = f(x)^2$;

3. Consider the differential equation $\frac{f'(x)-e^x}{f(x)} = 2$.
   a) Is this differential equation linear? Is there a way we can modify this differential equation to make it linear?
   b) Find one solution to the linear differential equation in question.
   c) What is the linear operator in question? What is its kernel?
   d) Find some solution of this differential equation which satisfies $f(0) = \frac{1}{2}$.

4. Let V and V' be two isomorphic vector spaces, and let T be the isomorphism from V to V'.
   a) Explain why T must be invertible. Let T^{-1} denote the inverse of T.
   b) Let S be a linear map from V to some vector space W. Construct a linear map S' from V' to W.
   c) Explain why S' will have the same rank and nullity as S.
   d) Let R be a linear map from W to V. Construct a linear map R' from W to V'.
   e) Explain why R' will have the same rank and nullity as R.

5. Let T_n be the linear differential operator which sends $f(x)$ to the n^th derivative of $f(x)$.
   a) Describe K_n, the kernel of T_n. What is its dimension?
   b) Explain why the kernel of the differential operator which sends f to $f' - f$ is isomorphic to K_1.
   c) To which K_n is the kernel of the differential operator which sends f to $2f' + 5f$ isomorphic?
   d) How about the one that sends f to $3f$?
   e) How about the one that sends f to $f'' + f$?
   f) For any linear differential operator S, the kernel of S will be isomorphic to some K_n. Which K_n will this be?

Solutions

1. a) Yes; the isomorphism is the transpose.
   b) No, they have different dimensions so are not isomorphic.
   c) Yes; the isomorphism sends c to $f(x) = ce^{-\frac{1}{6}x}$.
   d) No.
   e) Yes; the isomorphism sends (x, y) to $x + iy$.
   f) Yes; the isomorphism sends (x, y, z) to (e^x, e^y, e^z).

2. a) linear; b) not linear; c) linear; d) not linear;

3. a) The equation is not linear, but it can be rearranged as $f'(x) - 2f(x) = e^x$ which is linear.
   b) $f(x) = -e^x$ is one solution.
   c) The linear operator T sends f to $f' - 2f$. Its kernel is all functions of the form $f(x) = ce^{2x}$ for $c \in \mathbb{R}$.
   d) $f(x) = \frac{3}{2}e^{2x} - e^x$

4. a) T must be invertible because it is both injective and surjective and thus associates every element of V with exactly one element of V'.
   b) S' sends v to $S(T^{-1}(v))$.
   c) S' will have the same rank as S because it has the same range, and will have the same nullity because its kernel is isomorphic (via T) to the kernel of S.
   d) R' sends w to $T(R(w))$.
   e) R' will have the same rank as R because its range will be isomorphic (via T) to the range of R; it will have the same nullity because the kernels will be the same.

5. a) K_n will consist of all polynomials of degree $(n - 1)$. It will be n-dimensional.
   b) The differential operator which sends f to $f' - f$ is isomorphic to K_1 because it is also one-dimensional.
   c) This differential operator has a kernel isomorphic to K_2.
   d) K_0
   e) K_2
   f) For any linear differential operator S, the kernel of S will be isomorphic to K_n, where n is the degree of the highest derivative that appears in S.