COUNTING COVERS OF AN ELLIPTIC CURVE

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ABSTRACT. This note is an exposition of part of Dijkgraaf's article [Dij] on counting covers of elliptic curves and their connection with modular forms.

CONTENTS

1. STATEMENT OF THE PROBLEM

1.1. Fix an elliptic curve E, and a set $S = \{b_1, \ldots, b_{2g-2}\}$ of $2g-2$ distinct points of E. We define a *degree* d, *genus* g, *cover of* E to be an irreducible smooth curve C of genus g along with a finite degree d map $p : C \longrightarrow E$ simply branched over the points $b_1, \ldots, b_{2g-2}$ in S.

We consider two such covers $p_1 : C_1 \longrightarrow E$ and $p_2 : C_2 \longrightarrow E$ to be equivalent if they are isomorphic as schemes over E, that is, if there is an isomorphism $\phi : C_1 \xrightarrow{\sim} C_2$ commuting with the structure maps p_i to E.

For any cover $p : C \longrightarrow E$ we define the *automorphism group of the cover*, $\mathrm{Aut}_p(C)$ to be the automorphism group of C as a scheme over E, that is, the group

$$\mathrm{Aut}_p(C) := \{\phi : C \xrightarrow{\sim} C \mid p \circ \phi = p\}.$$

$\mathrm{Aut}_p(C)$ is always a finite group. We will usually abuse notation and write $\mathrm{Aut}(C)$ for this group, with the understanding that it depends on the structure map p to E.

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1.2. Let $\mathrm{Cov}(E,S)^\circ_{g,d}$ be the set of degree d, genus g covers of E, up to equivalence. We will refer to an element of $\mathrm{Cov}(E,S)^\circ_{g,d}$ by a representative (C,p) of the equivalence class.

If (C,p) is a cover, and (C',p') any cover equivalent to it, then $|\operatorname{Aut}_p(C)| = |\operatorname{Aut}_{p'}(C')|$. For any element of $\mathrm{Cov}(E,S)^\circ_{g,d}$ we assign it the weight $1/|\operatorname{Aut}_p(C)|$, where (C,p) is any representative of the class. This is well defined by the previous remark.

1.3. Let $N_{g,d}$ be the weighted count of the elements of $\mathrm{Cov}(E,S)^\circ_{g,d}$, each equivalence class being weighted as above.

The number $N_{g,d}$ is purely topological, and does not depend on the particular elliptic curve E chosen, nor on the set S of $2g-2$ distinct points, and we therefore omit them from the notation.

The first goal of this note is to explain how to calculate $N_{g,d}$ for all g and d; the second to link the generating functions to quasimodular forms, which we now describe.

1.4. For any even integer $k \geq 2$ define the *Eisenstein series of weight* k to be the series

$$E_k := 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n,$$

where B_k is the k-th Bernoulli number, and $\sigma_{k-1}(n)$ the sum $\sigma_{k-1}(n) := \sum_{m|n} m^{(k-1)}$.

The first three Eisenstein series are

$$\begin{aligned} E_2 &= 1 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^4 - \cdots, \\ E_4 &= 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \cdots, \quad \text{and} \\ E_6 &= 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 - \cdots. \end{aligned}$$

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ be the full modular group. If we set $q = \exp(2\pi i \tau)$, each E_k becomes a holomorphic function on the upper half plane, which is also, by virtue of its defining q-expansion, holomorphic at infinity. For $k \geq 4$ each E_k is a modular form of weight k.

1.5. The series E_2 is not modular. For any $\gamma \in \Gamma$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

(1.5.1) $$E_2(\gamma \cdot \tau) = (c\tau + d)^2 E_2(\tau) + \frac{6c(c\tau+d)}{\pi i}.$$

This can be easily computed using the identity

$$E_2(q) = \frac{1}{2\pi i}\frac{d}{d\tau}\log(\Delta(q)),$$

where $\Delta(q)$ is the weight 12 cusp form

$$\Delta(q) := q \prod_{n\geq 1}(1-q^n)^{24}.$$

1.6. Let $\Im(\tau)$ be the imaginary part of τ and $Y(\tau)$ the function $Y(\tau) = 4\pi\Im(\tau)$.

Kaneko and Zagier ([KZ], p. 166) define an *almost holomorphic modular form of weight k* to be a function $F(\tau)$ on the upper half plane of the form

$$F(\tau) = \sum_{m=0}^{M} f_m(\tau) Y^{-m}$$

where each $f_m(\tau)$ is holomorphic and grows at most polynomially in $1/Y$ as $Y \to 0$. In addition, $F(\tau)$ must satisfy the usual weight k modular transformation rule

$$F(\gamma \cdot \tau) = (c\tau + d)^k F(\tau).$$

They define a *quasimodular form of weight k* to be any holomorphic function $f_0(\tau)$ appearing as the "constant term with respect to $1/Y$" in such an expansion.

Since

$$\frac{1}{Y(\gamma \cdot \tau)} = \frac{(c\tau + d)^2}{Y(\tau)} + \frac{c(c\tau + d)}{2\pi i},$$

equation 1.5.1 shows that

$$E_2^* := E_2 - \frac{12}{Y}$$

is an almost holomorphic modular form of weight 2, and therefore that E_2 is a quasimodular form of the same weight.

1.7. Kaneko and Zagier ([KZ], p. 166) further define $\widetilde{M}_k(\Gamma)$ to be the vector space of quasimodular forms of weight k, and $\widetilde{M}(\Gamma) = \oplus_k \widetilde{M}_k(\Gamma)$ to be the graded ring of quasimodular forms.

They prove ([KZ], p. 167, Proposition 1) that $\widetilde{M}(\Gamma) = \mathbb{C}[E_2, E_4, E_6]$.

The monomials of weight k in E_2, E_4 and E_6 are linearly independent over $\mathbb{C}$ as functions on the upper half plane. One way to see this is to use the fact that monomials in E_4 and E_6 are modular forms, and linearly independent, and then apply equation 1.5.1 to any supposed linear relation among monomials involving E_2.

It follows that

$$\dim_{\mathbb{C}}(\widetilde{M}_k(\Gamma)) = \begin{cases} \left\lfloor \frac{(k+6)^2}{48} \right\rfloor & \text{if } k \not\equiv 0 \mod 12 \\ \\ \left\lfloor \frac{(k+6)^2}{48} \right\rfloor + 1 & \text{if } k \equiv 0 \mod 12 \end{cases}$$

or just $\dim_{\mathbb{C}}(\widetilde{M}_k(\Gamma)) = \left\lfloor \dfrac{(k+6)^2 + 12}{48} \right\rfloor$ for any even k.

1.8. For any $g \geq 1$, let

$$F_g(q) := \sum_{d \geq 1} N_{g,d}\ q^d$$

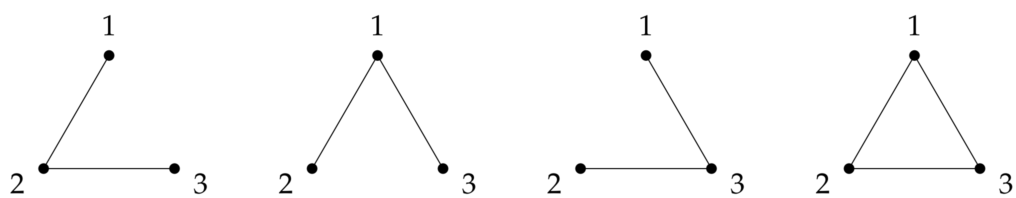
be the generating series counting covers of genus g. The interest in counting covers is because of the following surprising result of Dijkgraaf's.

Theorem (Dijkgraaf). — For $g \geq 2$, $F_g(q)$ is a quasimodular form of weight $6g - 6$.

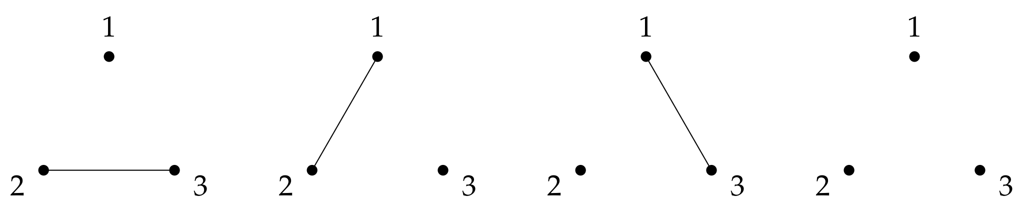
2. From connected to disconnected covers

2.1. A general feature of combinatorial problems is that it is often easier to count something if we remove the restriction that the objects in question be connected.

A standard example is to count the number of graphs with n labelled vertices. If $n = 3$ then there are four connected graphs with 3 labelled vertices:



If we allow (possibly) disconnected graphs, then there are a total of eight such graphs; the four above, and an additional four labelled graphs:



Let c_n be the number of connected graphs on n vertices, and d_n the (possibly) disconnected ones. It may not be clear how to calculate c_n, but since each graph on n vertices has a total of $\binom{n}{2}$ possible edges which we can either draw or omit, we have $d_n = 2^{\binom{n}{2}}$.

2.2. Suppose that we did know the number of connected graphs on n vertices. We would then have a second way to compute the number of possibly disconnected graphs. For instance, knowing that $c_1 = 1$, $c_2 = 2$, and $c_3 = 4$ we can compute that

$$d_3 = \binom{3}{3} c_3 + \binom{3}{2,1} c_2 c_1 + \frac{1}{3!} \binom{3}{1,1,1} c_1^3 = 8.$$

In this calculation, the sum is over the number of ways to break a graph with $n = 3$ vertices into connected pieces. In each term of the sum the binomial symbol (and factorial term) keeps track of the number of ways to split the vertices into connected pieces of the appropriate size. Finally the monomial in the c's calculates the number of such graphs given that particular division of the vertex set.

2.3. If we set

$$C(\lambda) := \sum_{n\geq 1} \frac{c_n}{n!}\lambda^n,$$

and

$$D(\lambda) := \sum_{n\geq 1} \frac{d_n}{n!}\lambda^n,$$

then our method of calculating the d_n's from the c_n's shows that

$$D(\lambda) = \exp(C(\lambda)) - 1.$$

This follows since the coefficient of λ^n in $\exp(C(\lambda))$ is clearly a sum of monomials in the c's, and the $n!$ terms in the definition of C and D, as well as the factorials appearing in the exponential ensure that these monomials appear with the correct coefficients. Working out the coefficient of λ^3 in $\exp(C(\lambda))$ is a good way to see how this happens.

Thanks to this identity, it is a formal matter to compute the d_n's if we know the c_n's, or conversely, by taking the logarithm, we can compute the c_n's from the d_n's. Since we do know a formula for the d_n's, this gives us a method to compute c_n for all n.

2.4. A typical strategy for attacking a combinatorial problem is the one used above: first solve the disconnected version of the problem, and then use that to compute the numbers for the connected case.

The process of passing from the connected to the possibly disconnected version of a combinatorial problem usually involves taking the exponential of the generating function. Describing a disconnected version generally requires partitioning some of the data in the problem (like the vertices in our example); the factorials and exponential take into account the combinatorics of making this division.

In order to apply this strategy to our counting problem, we first need to define the disconnected version of a cover.

2.5. Returning to the setup of section 1.1, let E be an elliptic curve and $S = \{b_1, \ldots, b_{2g-2}\}$ a set of $2g-2$ distinct points of E. We define a *degree d, genus g, disconnected cover of E* to be a union $C = \cup_i C_i$ of smooth irreducible curves, along with a finite degree d map $p : C \longrightarrow E$ simply branched over the points $b_1, \ldots, b_{2g-2}$ in S.

The condition on branching means that each ramification point of p is a simple ramification point, and that there are a total of $2g-2$ ramification points, mapped bijectively under p to $b_1, \ldots, b_{2g-2}$.

Note that by "disconnected cover" we mean disconnected in the weak sense: we allow the possibility that C is connected, so that this includes the case of degree d genus g

covers from section 1.1. A more accurate but also more clumsy name would be "possibly disconnected cover".

The restriction $p_i := p|_{C_i}$ of p to any component C_i of C is a finite map of some degree d_i. If C_i of C is of genus g_i, then by the Riemann-Hurwitz formula the map p_i will have $2g_i - 2$ ramification points on C_i. We therefore have the relations

$$\sum_i d_i = d, \text{ and } \sum_i (2g_i - 2) = 2g - 2.$$

2.6. As before, we define two covers to be equivalent if they are isomorphic over E. We define the the automorphism group of a cover $\mathrm{Aut}_p(C)$ to be the automorphism group of C as a scheme over E:

$$\mathrm{Aut}_p(C) := \{\phi : C \xrightarrow{\sim} C \mid p \circ \phi = p\}.$$

If C has no components of genus $g_i = 1$, then the automorphism group of the cover is the direct product of the automorphism group of the components.

$$\mathrm{Aut}_p(C) = \prod_i \mathrm{Aut}_{p_i}(C_i).$$

This follows from the fact that in this case no two components can be isomorphic as curves over E since they have different branch points over E.

If C has components of genus $g_i = 1$, then the map p_i has no ramification points and it is possible that some genus 1 components are isomorphic over E. In this case the automorphism group of the cover is the semidirect product of $\prod_i \mathrm{Aut}_{p_i}(C_i)$ and the permutations of the genus 1 components isomorphic over E.

2.7. Let $\mathrm{Cov}(E, S)_{g,d}$ be the set of genus g, degree d disconnected covers of E up to equivalence. We will refer to an element of $\mathrm{Cov}(E, S)_{g,d}$ by a representative (C, p) of the equivalence class.

For any element (C, p) of $\mathrm{Cov}(E, S)_{g,d}$, we give it the weight $1/|\mathrm{Aut}_p(C)|$. As in section 1.2, this is well defined.

2.8. Let $\widehat{N}_{g,d}$ be the number of genus g, degree d disconnected covers counted with the weighting above. As before, the number $\widehat{N}_{g,d}$ is purely topological and does not depend on the curve E or the particular distinct branch points $b_1, \ldots, b_{2g-2}$ chosen.

Let $Z(q, \lambda)$ be the generating function for the $N_{g,d}$'s:

$$
\begin{aligned}
Z(q,\lambda) \quad &:= \quad \sum_{g\geq 1}\sum_{d\geq 1} \frac{N_{g,d}}{(2g-2)!}\, q^d \lambda^{(2g-2)} \\
&= \quad \sum_{g\geq 1} \frac{F_g(q)}{(2g-2)!}\, \lambda^{(2g-2)},
\end{aligned}
$$

and $\widehat{Z}(q,\lambda)$ the corresponding generating function for the $\widehat{N}_{g,d}$'s:

$$
\widehat{Z}(q,\lambda) := \sum_{g\geq 1}\sum_{d\geq 1} \frac{\widehat{N}_{g,d}}{(2g-2)!}\, q^d \lambda^{(2g-2)}.
$$

2.9. *Lemma.* — The generating functions are related by $\widehat{Z}(q,\lambda) = \exp(Z(q,\lambda)) - 1$.

Proof. Let us organize the data of a disconnected cover by keeping track of the genus of each component, the degree of the map restricted to that component, and the number of times that each such pair occurs. We will call such data the *combinatorial type* of the cover.

Suppose that in a particular genus g, degree d, disconnected cover there are k_1 components of type (g_1, d_1), k_2 components of type (g_2, d_2), ..., and k_r components of type (g_r, d_r).

The numbers $\{k_j\}$, $\{g_j\}$, $\{d_j\}$, g and d satisfy the relations

$$
\sum_{j=1}^{r} k_j d_j = d, \text{ and } \sum_{j=1}^{r} k_j(2g_j - 2) = 2g - 2.
$$

We want to compute the weighted count of all disconnected genus g degree d covers of this combinatorial type. Clearly this number is some multiple of

$$
N^{k_1}_{g_1,d_1} N^{k_2}_{g_2,d_2} \cdots N^{k_r}_{g_r,d_r},
$$

the multiple depending on some combinatorial choices of branch points and some accounting for the automorphisms among the genus 1 components. We want to compute this multiple and see that it is the same as the coefficient of $N^{k_1}_{g_1,d_1} \cdots N^{k_r}_{g_r,d_r}$ appearing in $(2g-2)!\exp(Z(q,\lambda))$.

Since $\widehat{N}_{g,d}$ is the sum over the weighted counts of such topological types, and since the factor $(2g-2)!$ is included in the definition of $\widehat{Z}(q,\lambda)$, it will follow that $\widehat{Z}(q,\lambda) = \exp(Z(q,\lambda)) - 1$.

We first need to split up the $2g-2$ branch points into k_1 sets of $2g_1 - 2$, k_2 sets of $2g_2 - 2$, ..., and k_r sets of $2g_r - 2$ branch points. There are

$$
\binom{2g-2}{(2g_1-2), (2g_1-2), \ldots, (2g_r-2)} \prod_{k_j \text{ such that } g_j > 1} \frac{1}{k_j!}
$$

ways to make such a choice. Here, in the binomial symbol, each $2g_j - 2$ appears k_j times.

The binomial symbol alone counts the number of ways to split the $2g-2$ points into a set of $2g_1-2$ points, a second set of $2g_1-2$ points, ..., a k_r-th set of $2g_r-2$ points. However there is no natural order among sets of the same size which correspond to maps of the same degree (which is the "first" component of genus g_j and degree d_j?). Therefore we need to multiply by $1/k_j!$ for each k_j with $g_j > 1$, i.e., those k_j for which there are branch points, to account for this symmetrization.

From the description of the automorphism group in section 2.6 we see that the monomial $N^{k_1}_{g_1,d_1} \cdots N^{k_r}_{g_r,d_r}$ already takes care of most of the weighting coming from the automorphism group of the reducible cover. What is left is to account for the automorphisms coming from permuting genus 1 components isomorphic over E, as well as taking care of some overcounting in the monomial involving genus 1 components.

Suppose that $g_j = 1$ for some j, and that (C, p) is a particular genus g, degree d cover of the combinatorial type we are currently analyzing. Suppose that in our particular cover C, all of the components of type (g_j, d_j) are isomorphic over E. Then the automorphism group $\mathrm{Aut}_p(C)$ includes permutations of these components, automorphisms which are not taken care of by the monomial. We should therefore multiply by $1/k_j!$ to include this automorphism factor.

At the other extreme, suppose that in our particular cover C none of the components of type (g_j, d_j) are isomorphic over E. Then the claim is that the monomial $N^{k_1}_{g_1,d_1} \cdots N^{k_r}_{g_r,d_r}$ overcounts this cover by a factor of $k_j!$. Indeed, the product $N_{g_j,d_j} \cdot N_{g_j,d_j} \cdots N_{g_j,d_j} = N^{k_j}_{g_j,d_j}$ represents the act of choosing a genus g_j degree d_j cover k_j times. Since the resulting disjoint union of this components (as a scheme over E) will not depend on the order in which they are chosen, we have overcounted by a factor of $k_j!$ if the components are all distinct over E. Therefore in this case we should again multiply by $1/k_j!$ to compensate.

In the general case we should break the components of type (g_j, d_j) (still with $g_j = 1$) into subsets of components which are mutually isomorphic over E. The total contribution from permutation automorphisms and from correcting for overcounting is always $1/k_j!$.

Putting all this together, we see that the total contribution coming from covers of our particular combinatorial type is

$$\binom{2g-2}{(2g_1-2), (2g_1-2), \ldots, (2g_r-2)} \left(\prod_{j=1}^{r} \frac{1}{k_j!} \right) N^{k_1}_{g_1,d_1} \cdots N^{k_r}_{g_r,d_r}$$

$$= (2g-2)! \prod_{j=1}^{r} \left(\frac{N_{g_j,d_j}}{(2g_j-2)!} \right)^{k_j} \frac{1}{k_j!}.$$

Since this is exactly the expression appearing in $(2g-2)! \exp(Z(q,\lambda))$, the lemma is proved. □

2.10. Thanks to the lemma, we are now reduced to counting disconnected genus g, degree d covers. This problem is naturally equivalent to a counting problem in the symmetric group, by a method due to Hurwitz.

3. The monodromy map

3.1. Returning to the notation of section 2.5 let E be the elliptic curve and $S = \{b_1, \dots, b_{2g-2}\}$ the set of distinct branch points. Pick any point b_0 of E, $b_0 \notin S$.

We define a *marked disconnected cover* (or *marked cover* for short) of genus g and degree d to be a disconnected cover (C, p) of topological Euler characteristic $2 - 2g$ and degree d, along with a labelling from 1 to d of the d distinct points $p^{-1}(b_0)$.

We consider two marked covers (C, p), (C', p') to be equivalent if there is an isomorphism over E preserving the markings. Marked covers have no nontrivial automorphisms.

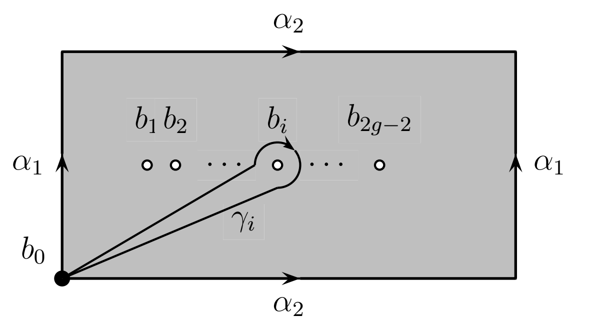
Let $\widetilde{\mathrm{Cov}}(E, S)_{g,d}$ be the set of marked covers up to equivalence. We will denote an element of $\widetilde{\mathrm{Cov}}(E, S)_{g,d}$ by $(\widetilde{C}, p)$.

3.2. Thanks to the marking, we have a natural monodromy map

$$\widetilde{\mathrm{Cov}}(E, S)_{g,d} \xrightarrow{\text{mon}} \mathrm{Hom}\,(\pi_1(E \setminus S, b_0), S_d)$$

where Hom is homomorphism of groups and S_d is the symmetric group on the set $\{1, \dots, d\}$. We wish to describe the image of this map.

Let $\pi_1 := \pi_1(E \setminus S, b_0)$. Cutting open the torus into a square with b_0 in the bottom left corner, the generators for the group π_1 are the loops α_1, α_2, and γ_i, $i = 1, \dots 2g-2$, where each γ_i is a small loop passing around b_i and returning to b_0.



The generators satisfy the single relation $\gamma_1\gamma_2 \cdots \gamma_{2g-2} = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}$.

Giving a homomorphism from π_1 to any group H is equivalent to giving elements $h_1, \dots, h_{2g-2}$, h, h' of H satisfying the relation $h_1 \cdots h_{2g-2} = hh'h^{-1}(h')^{-1}$.

Because of our conditions on the branching over b_i, $i = 1, \dots, 2g-2$, for any marked cover $(\widetilde{C}, p)$ the monodromy homomorphism sends γ_i to a simple transposition τ_i in S_d.

Let

(3.2.1) $$\widehat{T}_{g,d} := \left\{ (\tau_1, \tau_2, \dots, \tau_{2g-2}, \sigma_1, \sigma_2) \;\middle|\; \begin{array}{l} \tau_1, \dots, \tau_{2g-2}, \sigma_1, \sigma_2 \in S_d, \\ \text{each } \tau_i \text{ is a simple transposition,} \\ \tau_1\tau_2 \cdots \tau_{2g-2} = \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1} \end{array} \right\}.$$

The set $\widehat{T}_{g,d}$ classifies the homomorphisms $\mathrm{Hom}(\pi_1, S_d)$ of the type that can arise from covers in $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$, and so we get a classifying map $\rho : \widetilde{\mathrm{Cov}}(E,S)_{g,d} \longrightarrow \widehat{T}_{g,d}$ induced by monodromy.

3.3. *Lemma.* — The map $\rho : \widetilde{\mathrm{Cov}}(E,S)_{g,d} \longrightarrow \widehat{T}_{g,d}$ is surjective.

Proof. Suppose that t is an element of $\widehat{T}_{g,d}$ and $\psi_t : \pi_1 \longrightarrow S_d$ the corresponding homomorphism from π_1 to S_d. From the theory of covering spaces, this homomorphism gives rise to a finite étale map $p'' : C'' \longrightarrow (E \setminus S)^{\mathrm{AN}}$ along with a labelling of $(p')^{-1}(b_0)$ with monodromy equal to ψ_t. Here $(E \setminus S)^{\mathrm{AN}}$ is the analytic space associated to $(E \setminus S)$.

The theory of covering spaces only guarantees the existence of C'' as a complex manifold. However, by Grothendieck's version of the Riemann existence theorem ([SGA1], Exposé XII, Théorème 5.1) we may assume that $C'' = (C')^{\mathrm{AN}}$ and $p'' = (p')^{\mathrm{AN}}$ where C' is algebraic and $p' : C' \longrightarrow (E \setminus S)$ a finite étale map.

Let C'_i be any component of C' and p'_i the restriction of p' to C'_i. Let C_i be the integral closure of E in the function field of C'_i, and $p_i : C_i \longrightarrow E$ the associated map. Finally we set $\widetilde{C} = \cup_i C_i$ and define the map $p : \widetilde{C} \longrightarrow E$ to be the one given by p_i on each C_i.

Then $(\widetilde{C}, p)$ is an element of $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$ and $\rho((\widetilde{C}, p)) = t$. $\square$

3.4. The symmetric group S_d acts naturally on $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$. The natural map

$$\eta : \widetilde{\mathrm{Cov}}(E,S)_{g,d} \longrightarrow \mathrm{Cov}(E,S)_{g,d}$$

forgetting the marking makes $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$ a principal S_d set over $\mathrm{Cov}(E,S)_{g,d}$. The orbits of S_d on $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$ are therefore naturally in one to one correspondence with the elements of $\mathrm{Cov}(E,S)_{g,d}$.

The symmetric group S_d also acts on $\widehat{T}_{g,d}$ by conjugation; the map ρ is S_d-equivariant.

We will prove in lemma 3.6 that ρ is injective as a map from the set of S_d orbits of $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$ to the set of S_d orbits of $\widehat{T}_{g,d}$. Using this fact in conjunction with lemma 3.3, we will be able to conclude that the orbits of S_d on $\widehat{T}_{g,d}$ are also naturally in one to one correspondence with the elements of $\mathrm{Cov}(E,S)_{g,d}$.

For any element t of $\widehat{T}_{g,d}$ let $\mathrm{Stab}(t)$ be the stabilizer subgroup of t under the S_d action.

3.5. *Proposition.* — Let $(\widetilde{C}, p)$ be any element of $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$, $t = \rho((\widetilde{C}, p))$ the corresponding element of the classifying set $\widehat{T}_{g,d}$, and $C = \eta(\widetilde{C})$ the curve $\widetilde{C}$ with the markings forgotten. Then $\mathrm{Aut}_p(C) \cong \mathrm{Stab}(t)$.

Proof. Let $\mathrm{Fet}(E \setminus S)$ be the category of finite étale covers of $E \setminus S$. Each element of $\mathrm{Cov}(E,S)_{g,d}$ gives an element of $\mathrm{Fet}(E \setminus S)$, and the automorphisms of (C,p) as a cover over E are the same as the automorphisms of the corresponding object in $\mathrm{Fet}(E \setminus S)$.

By Grothendieck's theory of the algebraic fundamental group the fibre functor ("fibre over b_0") gives an equivalence of categories between $\mathrm{Fet}(E \setminus S)$ and the category of finite π_1 sets, i.e., the category of finite sets with π_1 action ([SGA1], Exposé V, Théorème 4.1).

More accurately, the theorem gives an equivalence of categories between $\mathrm{Fet}(E \setminus S)$ and the category of finite $\widehat{\pi}_1$ sets, where $\widehat{\pi}_1$ is the profinite completion of π_1, and the action of $\widehat{\pi}_1$ on finite sets is continuous. By the definition of profinite completion, this is the same as the category of finite π_1 sets.

Since we have an equivalence of categories, the automorphism group of an element of $\mathrm{Fet}(E \setminus S)$ is the same as the automorphism group of the associated π_1 set.

If D is a finite set with π_1 action, then an automorphism of D in the category of π_1 sets is a permutation $\sigma' \in S_D$ commuting with the π_1 action, where S_D is the permutation group of D. The action of π_1 on D is given by a homomorphism $\pi_1 \longrightarrow S_D$, and so the automorphism group of D as a π_1 set is just the group of elements in S_D commuting with the image of π_1. We may check if $\sigma' \in S_D$ commutes with the entire image of π_1 by checking if it commutes with the images of the generators.

Since $t \in \widehat{T}_{g,d}$ is the list of images of the generators of π_1 on the fibre $D := p^{-1}(b_0) \cong \{1, \ldots, d\}$ of C, and since for σ' to commute with a generator is the same as saying that the action by conjugation of σ' on the generator is trivial, the proposition follows. $\square$

The equivalence of categories lets us clear up one point left over from section 3.4.

3.6. *Lemma.* — The map ρ is injective as a map from the set of S_d orbits on $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$ to the set of S_d orbits on $\widehat{T}_{g,d}$.

Proof. Suppose that $(\widetilde{C}, p)$ and $(\widetilde{C}', p')$ are two elements of $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$ whose images under ρ are in the same S_d orbit. By using the S_d action, we may in fact assume that they have the same image $t \in \widehat{T}_{g,d}$ under ρ. We want to show that $(\widetilde{C}, p)$ and $(\widetilde{C}', p')$ are in the same S_d orbit in $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$.

But, since the π_1 set associated to both $\widetilde{C}$ and $\widetilde{C}'$ is the same, it follows from the fact that the fibre functor defines an equivalence of categories that $\eta(\widetilde{C})$ and $\eta(\widetilde{C}')$ are isomorphic as objects of $\mathrm{Fet}(E \setminus S)$, and therefore that $\eta(\widetilde{C})$ and $\eta(\widetilde{C}')$ are isomorphic over E, and so (by the definition of $\mathrm{Cov}(E,S)_{g,d}$) that $\eta(\widetilde{C})$ and $\eta(\widetilde{C}')$ are the same element of $\mathrm{Cov}(E,S)_{g,d}$.

Since $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$ is a a principal S_d set over $\mathrm{Cov}(E,S)_{g,d}$ via η, this implies that $(\widetilde{C}, p)$ and $(\widetilde{C}', p')$ are in the same S_d orbit, and therefore that ρ is an injective map between the sets of orbits. $\square$

3.7. Let G be a finite group and X a finite set with G action. For any $x \in X$ let G_x be the stabilizer subgroup of x. If x and x' are in the same G-orbit, then $G_x \cong G_{x'}$ and so $|G_x| = |G_{x'}|$.

We wish to compute a weighted sum of the orbits in X, with the following weighting: if $\mathcal{O}$ is any G-orbit in X, assign it the weighting $1/|G_x|$ where x is any element $x \in \mathcal{O}$. This is well defined by the previous remark.

3.8. *Lemma.* — Let G be a finite group acting on a finite set X. Then the weighted count of G-orbits of X is $|X|/|G|$. In particular, it is independent of the G action.

Proof. This follows immediately from the orbit-stabilizer theorem. $\square$

By lemma 3.6 and the discussion in section 3.4 the elements of $\mathrm{Cov}(E,S)_{g,d}$ are naturally in one to one correspondence with the S_d orbits on $\widehat{T}_{g,d}$. By proposition 3.5 the group $\mathrm{Aut}_p(C)$ is isomorphic to $\mathrm{Stab}(t)$ for any t in the orbit corresponding to (C,p). Applying lemma 3.8 and recalling the definition of $\widehat{N}_{g,d}$ from section 2.8 we have proved the following:

3.9. *Reduction Step I.* — $\widehat{N}_{g,d} = |\widehat{T}_{g,d}|/d!$.

Our counting problem has now been reduced to computing the size of $\widehat{T}_{g,d}$.

4. A calculation in the symmetric group

4.1. In order to compute the size of $\widehat{T}_{g,d}$ let us fix a σ_2 in S_d, and ask how many elements t of $\widehat{T}_{g,d}$ end with σ_2. This is easier to analyze if we rewrite the defining condition in 3.2.1 as

(4.1.1)
$$(\tau_1\tau_2\cdots\tau_{2g-2})\sigma_2 = \sigma_1\sigma_2\sigma_1^{-1},$$

from which we see that main obstacle is that $(\tau_1\cdots\tau_{2g-2})\sigma_2$ should be conjugate to σ_2.

For any $\sigma_2 \in S_d$, let

$$P_{g,d}(\sigma_2) := \left\{ (\tau_1,\tau_2,\ldots,\tau_{2g-2}) \;\middle|\; \begin{array}{l} \text{each } \tau_i \in S_d \text{ is a simple transposition,} \\ \text{and } (\tau_1\tau_2\cdots\tau_{2g-2})\sigma_2 \text{ is conjugate to } \sigma_2 \end{array} \right\}$$

By the definition of $P_{g,d}(\sigma_2)$, for any $(\tau_1,\ldots,\tau_{2g-2})$ in $P_{g,d}(\sigma_2)$ there exists at least one $\sigma_1 \in S_d$ with $\tau_1\ldots\tau_{2g-2}\sigma_2 = \sigma_1\sigma_2\sigma_1^{-1}$.

Once we have found one such σ_1, the others differ from it by an element commuting with σ_2. In other words, once we fix σ_2, and $(\tau_1,\ldots,\tau_{2g-2}) \in P_{g,d}(\sigma_2)$ there are exactly as many possibilities for σ_1 satisfying 4.1.1 as there are elements which commute with σ_2.

If $c(\sigma_2)$ is the conjugacy class of σ_2, and $|c(\sigma_2)|$ the size of $c(\sigma_2)$, then the number of elements commuting with σ_2 is $|S_d|/|c(\sigma_2)|$, or $d!/|c(\sigma_2)|$ and so we get the formula

$$|\widehat{T}_{g,d}| = \sum_{\sigma_2 \in S_d} \frac{d!}{|c(\sigma_2)|} |P_{g,d}(\sigma_2)|.$$

4.2. In general, if c is any conjugacy class of S_d, let $|c|$ denote the size of c. If $f : S_d \longrightarrow \mathbb{C}$ is any function from S_d to $\mathbb{C}$ which is constant on conjugacy classes, then let $f(c)$ denote the value of f on any element $\sigma \in c$ of that conjugacy class. Finally, let c^{-1} denote the conjugacy class made up of inverses of elements in c.

For any $\sigma, \sigma_2 \in S_d$ we have $P_{g,d}(\sigma\sigma_2\sigma^{-1}) = \sigma P_{g,d}(\sigma_2)\sigma^{-1}$ where the conjugation of the set $P_{g,d}(\sigma_2)$ means to conjugate the entries of all elements in $P_{g,d}(\sigma_2)$. This calculation shows that the function $|P_{g,d}(\,\cdot\,)| : S_d \longrightarrow \mathbb{N}$ is constant on conjugacy classes.

Using our notational conventions, we can now write the previous formula as

$$|\widehat{T}_{g,d}| = \sum_{c} \sum_{\sigma_2 \in c} \frac{d!}{|c|} |P_{g,d}(c)| = \sum_{c} |c| \frac{d!}{|c|} |P_{g,d}(c)| = \sum_{c} d! |P_{g,d}(c)|$$

where the sum $\sum_c$ means to sum over the conjugacy classes of S_d.

On account of reduction 3.9 we then have

$$\widehat{N}_{g,d} = \sum_{c} |P_{g,d}(c)|. \tag{4.2.1}$$

4.3. The problem of computing $|P_{g,d}(c)|$ looks very much like the problem of computing the number of cycles in a graph.

Imagine a graph where the vertices are indexed by the conjugacy classes c of S_d, and the edges are the conjugacy classes which can be connected by multiplying by a transposition. For any conjugacy class c, the number $|P_{g,d}(c)|$ then looks like an enumeration of the paths of length $2g-2$ starting and ending at vertex c.

This picture is not quite correct since there is fundamental asymmetry in the definition of $P_{g,d}$ – we are really only allowed to pick a particular representative σ_2 of the conjugacy class c, and ask for "paths" (i.e. sequences of transpositions) which join σ_2 to c.

If we take this asymmetry into account we can define a version of the adjacency matrix which will allow us compute $|P_{g,d}(c)|$ just as in the case of graphs.

4.4. For any $d \geq 1$ let M_d be the square matrix whose rows and columns are indexed by the conjugacy classes of S_d.

In the column indexed by the conjugacy class c and the row indexed by the conjugacy class c' we define the entry $(M_d)_{c',c}$ as follows: Pick any representative σ_2 of c, and let $M_{c',c}$

be the number of transpositions $\tau \in S_d$ such that $\tau\sigma_2 \in c'$. This number is independent of the representative $\sigma_2 \in c$ picked.

As an example, here are the matrices for $d = 3$ and $d = 4$, with the conjugacy classes represented by the partitioning of the dots.

$$\begin{array}{c|ccc} & \bullet\bullet\bullet & \bullet(\bullet\bullet) & (\bullet\bullet\bullet) \\ \hline \bullet\bullet\bullet & 0 & 1 & 0 \\ \bullet(\bullet\bullet) & 3 & 0 & 3 \\ (\bullet\bullet\bullet) & 0 & 2 & 0 \end{array}$$

M_3

$$\begin{array}{c|ccccc} & \bullet\bullet\bullet\bullet & \bullet\bullet(\bullet\bullet) & \bullet(\bullet\bullet\bullet) & (\bullet\bullet)(\bullet\bullet) & (\bullet\bullet\bullet\bullet) \\ \hline \bullet\bullet\bullet\bullet & 0 & 1 & 0 & 0 & 0 \\ \bullet\bullet(\bullet\bullet) & 6 & 0 & 3 & 2 & 0 \\ \bullet(\bullet\bullet\bullet) & 0 & 4 & 0 & 0 & 4 \\ (\bullet\bullet)(\bullet\bullet) & 0 & 1 & 0 & 0 & 2 \\ (\bullet\bullet\bullet\bullet) & 0 & 0 & 3 & 4 & 0 \end{array}$$

M_4

4.5. *Lemma.* — For any $k \geq 1$, the entry in column c and row c' of M_d^k is given by:

$$(M_d^k)_{c',c} = \left| \left\{ (\tau_1, \ldots, \tau_k) \;\middle|\; \begin{array}{l} \text{Each } \tau_i \in S_d \text{ is a transposition,} \\ \text{and } (\tau_1 \cdots \tau_k)\sigma_2 \in c' \end{array} \right\} \right|$$

where σ_2 is any element $\sigma_2 \in c$. The calculation of this number does not depend on the representative σ_2 chosen.

Proof. Straightforward induction argument. The case $k = 1$ is the definition of M_d. □

Lemma 4.5 has the following useful corollary.

4.5.1. *Corollary.* —

(a) If k is odd, then $(M_d^k)_{c,c} = 0$ for all conjugacy classes c.
(b) If $k = 2g - 2$ is even, then $(M_d^{2g-2})_{c,c} = |P_{g,d}(c)|$ for all conjugacy classes c.

Proof. Part (a) follows from the lemma and parity considerations – an odd number of transpositions can never take an element of a conjugacy class back to the same class (or to any other class of the same parity).

Part (b) follows immediately from the lemma and the definition of $|P_{g,d}(c)|$. □

4.6. Applying part (b) of the corollary and equation 4.2.1 we now have

$$\widehat{N}_{g,d} = \mathrm{Tr}(M_d^{2g-2}).$$

where Tr is the trace of a matrix.

Along with part (a) of the corollary and the definition (section 2.8) of $\widehat{Z}(q,\lambda)$ this gives

$$\textbf{(4.6.1)} \qquad \widehat{Z}(q,\lambda) = \sum_{g\geq 1}\sum_{d\geq 1} \frac{\mathrm{Tr}(M_d^{2g-2})}{(2g-2)!}\, q^d \lambda^{(2g-2)} = \sum_{d\geq 1} \mathrm{Tr}(\exp(M_d\cdot\lambda))\, q^d.$$

4.7. For any integer $d \geq 1$, let $\mathrm{part}(d)$ be the number of partitions of the integer d. This is exactly the number of conjugacy classes in S_d, so that each M_d is a $\mathrm{part}(d)\times\mathrm{part}(d)$ matrix.

For each d, let $\{\mu_{i,d}\}$, $i = 1, \ldots, \mathrm{part}(d)$, be the eigenvalues of M_d. Since

$$\mathrm{Tr}(M_d^k) = \sum_{i=1}^{\mathrm{part}(d)} \mu_{i,d}^k,$$

we can also write equation 4.6.1 as

4.8. *Reduction Step II.* — $\widehat{Z}(q,\lambda) = \sum_{d\geq 1}\sum_{i=1}^{\mathrm{part}(d)} \exp(\mu_{i,d}\lambda)$.

Therefore we have reduced our counting problem yet again – this time to computing the eigenvalues of the matrices M_d.

5. A calculation in the group algebra

5.1. Let $\mathbb{C}[S_d]$ be the group algebra of S_d, and $\mathcal{H}_d$ its center. For any conjugacy class c of S_d, let

$$z_c := \sum_{\sigma\in c} \sigma$$

be the sum in $\mathbb{C}[S_d]$ of the group elements in the class c. It is well known that the $\{z_c\}$ form a basis for the center $\mathcal{H}_d$ and therefore that $\mathcal{H}_d$ is a $\mathrm{part}(d)$ dimensional $\mathbb{C}$-algebra.

Since $\mathcal{H}_d$ is an algebra, multiplication by any element $z \in \mathcal{H}_d$ is a $\mathbb{C}$-linear map, and can be represented by a $\mathrm{part}(d)\times\mathrm{part}(d)$ matrix.

Let τ stand for the conjugacy class of a transposition, and z_τ the corresponding basis element of $\mathcal{H}_d$. In the $\{z_c\}$ basis, the matrices for multiplication by z_τ in $\mathcal{H}_3$ and $\mathcal{H}_4$ are

$$\begin{array}{c c} & \begin{array}{ccc} \bullet\bullet\bullet & \bullet(\bullet\bullet) & (\bullet\bullet\bullet) \end{array} \\ \begin{array}{c} \bullet\bullet\bullet \\ \bullet(\bullet\bullet) \\ (\bullet\bullet\bullet) \end{array} & \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix} \end{array} \qquad \text{and} \qquad \begin{array}{c c} & \begin{array}{ccccc} \bullet\bullet\bullet\bullet & \bullet\bullet(\bullet\bullet) & \bullet(\bullet\bullet\bullet) & (\bullet\bullet)(\bullet\bullet) & (\bullet\bullet\bullet\bullet) \end{array} \\ \begin{array}{c} \bullet\bullet\bullet\bullet \\ \bullet\bullet(\bullet\bullet) \\ \bullet(\bullet\bullet\bullet) \\ (\bullet\bullet)(\bullet\bullet) \\ (\bullet\bullet\bullet\bullet) \end{array} & \begin{bmatrix} 0 & 6 & 0 & 0 & 0 \\ 1 & 0 & 4 & 1 & 0 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 4 \\ 0 & 0 & 4 & 2 & 0 \end{bmatrix} \end{array}$$

in $\mathcal{H}_3$ $\qquad\qquad$ in $\mathcal{H}_4$.

5.2. *Proposition.* — In the $\{z_c\}$ basis, the matrix for multiplication by z_τ in $\mathcal{H}_d$ is the transpose of M_d.

Proof. The entry in column c' and row c of the multiplication matrix is the coefficient of z_c in the expansion of $z_\tau \cdot z_{c'}$ into basis vectors.

To compute this, pick any $\sigma_2 \in c$, and ask how many times σ_2 appears in the product expansion

$$z_\tau \cdot z_{c'} = \left(\sum_{\tau_i \in \tau} \tau_j\right) \cdot \left(\sum_{\sigma'_j \in c'} \sigma'_j\right).$$

This is clearly the number of times that there exists a $\tau_i \in \tau$ and $\sigma'_j \in c'$ with $\tau_i \sigma'_j = \sigma_2$, which is perhaps more easily phrased as the number of times that there is a transposition $\tau_i \in \tau$ with $\tau_i^{-1}\sigma_2 \in c'$.

Since the inverse of a transposition is a transposition, we see that this is the same as the entry $(M_d)_{c',c}$ in column c and row c' of M_d, and therefore that the multiplication matrix is the transpose of M_d. □

It is therefore enough to understand the eigenvalues for z_τ acting on $\mathcal{H}_d$ by multiplication.

5.3. Since $\mathcal{H}_d$ is a commutative algebra, for any action of $\mathcal{H}_d$ on a finite dimensional vector space, we might hope to diagonalize the action by finding a basis of simultaneous eigenvectors. In particular, we might hope to find such a basis for $\mathcal{H}_d$ acting on itself.

5.4. Let χ be an irreducible character of S_d, and define

$$w_\chi := \frac{\dim(\chi)}{d!} \sum_c \chi(c^{-1})\ z_c = \frac{\dim(\chi)}{d!} \sum_{\sigma \in S_d} \chi(\sigma^{-1})\ \sigma,$$

where $\dim(\chi) = \chi(1)$ is the dimension of the irreducible representation associated to χ. Each w_χ is by definition in $\mathcal{H}_d$, and we will see below that the $\{w_\chi\}$ form a basis for $\mathcal{H}_d$.

There are two well known orthogonality formulas involving the characters. For the first, pick any σ_1 in S_d, then

(5.4.1)
$$\sum_{\sigma \in S_d} \chi(\sigma)\chi'(\sigma^{-1}\sigma_1) = \begin{cases} 0 & \text{if } \chi \neq \chi' \\ \frac{d!}{\dim(\chi)}\chi(\sigma_1) & \text{if } \chi = \chi' \end{cases}.$$

The second is an orthogonality between the conjugacy classes. For any conjugacy classes c and c',

(5.4.2)
$$\sum_\chi \chi(c)\chi(c') = \begin{cases} 0 & \text{if } c \neq c' \\ \frac{d!}{|c|} & \text{if } c = c' \end{cases},$$

where the sum is over the characters χ of S_d.

Formula 5.4.1 is equivalent to the multiplication formula

(5.4.3)
$$w_\chi \cdot w_{\chi'} = \begin{cases} 0 & \text{if } \chi \neq \chi' \\ w_\chi & \text{if } \chi = \chi' \end{cases},$$

while formula 5.4.2 is equivalent to the following expression for z_c in terms of the w_χ

(5.4.4)
$$z_c = \sum_\chi \left(\frac{|c^{-1}|\chi(c^{-1})}{\dim(\chi)} \right) w_\chi.$$

To see this last equation, expand the term on the right using the definition of w_χ

$$\sum_\chi \left(\frac{|c^{-1}|\chi(c^{-1})}{\dim(\chi)} \right) w_\chi = \sum_{\sigma \in S_d} \left(\sum_\chi \frac{|c^{-1}|}{d!} \chi(c^{-1})\chi(\sigma^{-1}) \right) \sigma = \sum_{\sigma \in c} \sigma = z_c;$$

the middle equality comes from applying 5.4.2.

Equation 5.4.4 shows that the $\{w_\chi\}$ span $\mathcal{H}_d$. Either formula 5.4.3 or the fact that the number of characters is the same as the number of conjugacy classes shows that the $\{w_\chi\}$ are linearly independent. The $\{w_\chi\}$ therefore form a basis for $\mathcal{H}_d$.

5.5. Formula 5.4.3 is the most important; it shows that the basis $\{w_\chi\}$ diagonalizes the action of $\mathcal{H}_d$ on itself by multiplication. In particular, if

$$z = \sum_\chi a_\chi w_\chi$$

is any element in $\mathcal{H}_d$, then the eigenvalues of the matrix for multiplication by z are precisely the coefficients a_χ of z expressed in the $\{w_\chi\}$ basis.

The coefficients of z_τ in the $\{w_\chi\}$ basis are given by equation 5.4.4 with $c = \tau$. Since $\tau^{-1} = \tau$ (the inverse of a transpose is a transpose), and since by proposition 5.2 the eigenvalues for multiplication by z_τ are the same as the eigenvalues of M_d, we have proven

5.6. *Reduction Step III.* — The eigenvalues $\{\mu_{i,d}\}$ of M_d are given by

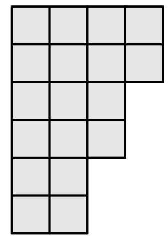
$$\frac{\binom{d}{2}\chi(\tau)}{\dim(\chi)}$$

as χ runs through the irreducible characters of S_d.

6. A FORMULA OF FROBENIUS

6.1. In order to use reduction 5.6 to calculate $\widehat{Z}(q,\lambda)$ we need to have a method to compute $\chi(\tau)/\dim(\chi)$ for the characters χ of S_d.

The irreducible representations of S_d are in one to one correspondence with the partitions of d. Any such partition can be represented by a *Young diagram*:

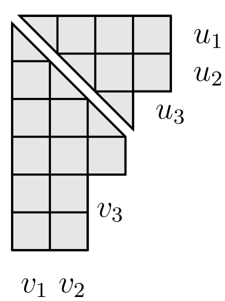


The picture represents the partition $4+4+3+3+2+2$ of $d = 18$.

There are many formulas relating the combinatorics of the Young diagram to the data of the irreducible representation associated to it. For example, the hooklength formula computes the dimension $\dim(\chi)$ of the irreducible representation, while the Murnaghan-Nakayama formula computes the value of the character on any conjugacy class.

More convenient for us is the following somewhat startling formula of Frobenius.

6.2. Given a Young diagram, split it diagonally into two pieces;



because the splitting is diagonal, there are as many rows in the top piece as columns in the bottom piece.

Suppose that there are r rows in the top and r columns in the bottom. Let u_i, $i = 1, \dots r$ be the number of boxes in the i-th row of the top piece, and v_i, $i = 1, \dots r$ the number of boxes in the i-th column of the bottom piece. The numbers u_i, v_i are *half integers*. In the example $r = 3$ and the numbers are $u_1 = 3\frac{1}{2}$, $u_2 = 2\frac{1}{2}$, $u_3 = \frac{1}{2}$ and $v_1 = 5\frac{1}{2}$, $v_2 = 4\frac{1}{2}$, $v_3 = 1\frac{1}{2}$.

If χ is the irreducible character associated to the partition, and τ the conjugacy class of transpositions in S_d, then Frobenius's formula is

$$\frac{\binom{d}{2}\chi(\tau)}{\dim(\chi)} = \frac{1}{2}\left(\sum_{i=1}^{r} u_i^2 - \sum_{i=1}^{r} v_i^2\right).$$

In our example this is

$$\frac{1}{2}\left((3\frac{1}{2})^2 + (2\frac{1}{2})^2 + (\frac{1}{2})^2 - (5\frac{1}{2})^2 - (4\frac{1}{2})^2 - (1\frac{1}{2})^2\right) = -17.$$

The formula produces exactly the eigenvalues we are looking for.

6.3. Let $\mathbb{Z}_{\geq 0+\frac{1}{2}}$ be the set

$$\mathbb{Z}_{\geq 0+\frac{1}{2}} := \left\{ \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{7}{2},\ \ldots\ldots \right\}$$

of positive half integers.

The process of cutting Young diagram diagonally gives a one to one correspondence between partitions and subsets U and V of $\mathbb{Z}_{\geq 0+\frac{1}{2}}$ with $|U| = |V|$.

To reverse it, given any two such subsets with $|U| = |V| = r$, we organize the elements $u_1, \ldots, u_r$ so that $u_1 > u_2 > \cdots > u_r$, and similarly with the v_i's. We then recover the Young diagram by gluing together the appropriate row with u_i boxes to the column with v_i boxes, $i = 1, \ldots, r$.

The resulting Young diagram is a partition of $d = \sum_{u \in U} u + \sum_{v \in V} v$.

Since the data of the subsets U and V are sufficient to recover the degree d, and the eigenvalue associated to the corresponding irreducible representation, we see that all of the combinatorial information we are interested in is contained in these subsets.

6.4. Consider the infinite product

$$\prod_{u \in \mathbb{Z}_{\geq 0+\frac{1}{2}}} \left(1 + \zeta\, q^u\, e^{\frac{u^2}{2}\lambda}\right) \prod_{v \in \mathbb{Z}_{\geq 0+\frac{1}{2}}} \left(1 + \zeta^{-1}\, q^v\, e^{\frac{-v^2}{2}\lambda}\right).$$

Computing a term in the expansion of this product involves choosing finite subsets $U \subset \mathbb{Z}_{\geq 0+\frac{1}{2}}$ and $V \subset \mathbb{Z}_{\geq 0+\frac{1}{2}}$.

In the term corresponding to a pair of subsets U and V,

- The power of ζ appearing in the term is $|U| - |V|$,
- the power of q appearing is $\sum_{u \in U} u + \sum_{v \in V} v$, and
- the exponential term is $\exp\left(\frac{1}{2}(\sum_{u \in U} u^2 - \sum_{v \in V} v^2)\lambda\right)$.

The infinite product is a Laurent series in ζ with coefficients formal power series in q and λ. Combining the above discussion, the formula of Frobenius, and reduction 4.8 we have

6.5. *Reduction Step IV.* —

$$\widehat{Z}(q, \lambda) = \text{coeff of } \zeta^0 \text{ in } \left(\prod_{u \in \mathbb{Z}_{\geq 0+\frac{1}{2}}} \left(1 + \zeta\, q^u\, e^{\frac{u^2}{2}\lambda}\right) \prod_{v \in \mathbb{Z}_{\geq 0+\frac{1}{2}}} \left(1 + \zeta^{-1}\, q^v\, e^{\frac{-v^2}{2}\lambda}\right) \right) - 1.$$

The "minus 1" is because $\widehat{Z}(q,\lambda)$ doesn't have a constant term, or alternately, because we should ignore the term where both U and V are the empty set.

The fact that $F_g(q)$ is a quasimodular form of weight $6g-6$ follows from this formula and the work of Kaneko and Zagier.

7. The work of Kaneko and Zagier

7.1. Kaneko and Zagier [KZ] start with the series

$$\Theta(q,\lambda,\zeta) := \prod_{n\geq 1}(1-q^n) \prod_{u\in\mathbb{Z}_{\geq 0+\frac{1}{2}}} \left(1+\zeta\, q^u\, e^{\frac{u^2}{2}\lambda}\right) \prod_{v\in\mathbb{Z}_{\geq 0+\frac{1}{2}}} \left(1+\zeta^{-1}\, q^v\, e^{\frac{-v^2}{2}\lambda}\right)$$

considered as a Laurent series in ζ with coefficients formal power series in q and λ.

Let $\Theta_0(q,\lambda)$ be the coefficient of ζ^0 in this series, and write $\Theta_0(q,\lambda) = \sum_k A_k(q)\lambda^k$.

They prove ([KZ], Theorem 1) that $A_k(q)$ is a quasimodular form of weight $3k$.

By reduction 6.5, $\Theta_0(q,\lambda) = (\prod_{n\geq 1}(1-q^n))(\widehat{Z}(q,\lambda)+1)$ and so taking the logarithm and using lemma 2.9 we get

$$\begin{aligned}\log(\Theta_0(q,\lambda)) &= \log\left(\prod_{n\geq 1}(1-q^n)\right) + \log(\widehat{Z}(q,\lambda)+1)\\ &= \sum_{n\geq 1}\log(1-q^n) + Z(q,\lambda)\end{aligned}$$

Using Kaneko and Zagier's theorem, the coefficient of λ^k in $\log(\Theta_0(q,\lambda))$ is also a quasi-modular form of weight $3k$. In particular, the coefficient of λ^{2g-2} is a quasimodular form of weight $6g-6$.

Since the $\log(1-q^n)$ terms contain no power of λ, as long as $g\geq 2$ the coefficient of λ^{2g-2} is $F_g(q)/(2g-2)!$, and therefore $F_g(q)$ is a quasimodular form of weight $6g-6$.

As for the λ^0 term, $F_1(q)$ is exactly equal to $-\sum_{n\geq 1}\log(1-q^n)$, and so this is cancelled out in the expression for $\log(\Theta_0(q,\lambda))$.

References

[Dij] R. Dijkgraaf *Mirror Symmetry and Elliptic Curves*, in *The moduli space of curves*, Progess in Mathematics, Volume 129, Birkhäuser 1995.

[SGA1] A. Grothendieck *Revetements Étales et Groupe Fondamental*, Springer Lecture Notes 224.

[KZ] M. Kaneko and D. Zagier, *A Generalized Jacobi Theta Function and Quasimodular Forms*, in *The moduli space of curves*, Progess in Mathematics, Volume 129, Birkhäuser 1995.