1. Write each of the following complex numbers in the form $a + bi$.

(a) $(2+3i)(3+4i)$, (b) $\dfrac{2+3i}{3+4i}$, (c) $\dfrac{1}{1-i} + \dfrac{1}{1+2i} + \dfrac{1}{3+i}$.

2.

(a) Use L'Hôpital's rule to show that for any integer $n \geq 0$, $\lim_{u\to\infty} \dfrac{u^n}{e^u} = 0$.

(b) Use a substitution and the result of (a) to prove that $\lim_{y\to\infty} \dfrac{y^n}{e^{y^2}} = 0$ for any integer $n > 0$.

Note that you may have to make a small extra argument when n is odd.

(c) Use a substitution and the result of (b) to prove the lemma from the first day of class: For any integer $n > 0$, $\lim_{x\to 0} \dfrac{e^{-\frac{1}{x^2}}}{x^n} = 0$.

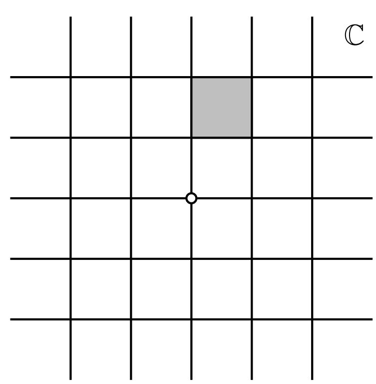
3. Suppose that $a \in \mathbb{C}$ is a fixed complex number. Find the maximum value of

$$\left\{ |z^n + a| \,\middle|\, |z| \leq 1, n \in \mathbb{Z}, n \geq 1 \right\}.$$

(The easiest argument probably involves a preliminary reduction step, and then geometry.)

4. To visualize a complex function it would be convenient if we could draw the graph. Since the graph is a subset of $\mathbb{C}^2$ (which has four real dimensions) this is not possible. The next best thing is to draw some shapes in $\mathbb{C}$ and try and visualize a function f by seeing what happens to those shapes when we apply f.

At right is a picture of the lines $\mathrm{Re}(z) \in \{-2,-1,0,1,2\}$ and $\mathrm{Im}(z) \in \{-2,-1,0,1,2\}$ in $\mathbb{C}$, with with the rectangle $\{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1, 1 \leq \mathrm{Im}(z) \leq 2\}$ shaded in.



Draw what happens to this picture under the maps

(a) $f(z) = z^2$,

(b) $f(z) = \frac{1}{z}$.

5. The purpose of this problem is to prove that the image of a circle under a Möbius transformation is always either a line or a circle.

Suppose that $f(x) = \frac{az+b}{cz+d}$ is a Möbius transformation (i.e, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$).

(a) Explain how you know that f is invertible, and give a formula for the inverse function $f^{-1}(z)$.

(b) Suppose that $p \in \mathbb{C}$, $r \in \mathbb{R}$ (with $r > 0$), and let C be the circle $C = \left\{ z \;\middle|\; |z - p| = r \right\}$. (i.e., C is the circle of radius r centred at p).

Explain why the image of C under the map f is the same as the set $\left\{ z \;\middle|\; |f^{-1}(z) - p| = r \right\}$.

(c) Use part (b) and a result from class to show that the image of C under f is a line or a circle.

(d) Let C be the circle of radius $\sqrt{5}$ centered at $2 - 2i$. Find the image of C under the map $f(z) = \frac{15}{z}$. In particular, find the center and radius of the new circle.

NOTE: In part (d), the center of $f(C)$ is *not* the point obtained by applying f to the center of C, nor is there any automatic way of computing the new radius. Möbius transformations take circles to circles (or lines), but they don't usually take the center of one circle to the center of another. Your best bet for (d) is to work directly with equations.