Math 326 — Homework Assignment 2

DUE DATE: SEPT. 30, 2010

1. Describe (and draw) the images of the following sets under the map $f(z) = \exp(z)$.

   (a) $S = \left\{ z \,\middle|\, \frac{\pi}{4} \le \mathrm{Im}(z) \le \frac{2\pi}{3} \right\}$ (b) $S = \left\{ z \,\middle|\, 1 \le \mathrm{Re}(z) \le 2 \right\}$.

2. Describe (and draw) the images of the following sets under the map $f(z) = \mathrm{Log}(z)$.

   (a) $S = \left\{ z \,\middle|\, \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0 \right\}$ (b) $S = \left\{ z \,\middle|\, |z| \ge e \right\}$.

NOTE: The function Log is the principal branch of log, based on the principal branch of arg (i.e, not the multivalued log). Note that the set in (b) has a boundary.

3. Find all values of

   (a) $(1+i)^i$, (b) $2^{\sqrt{2}}$, (c) $i^{\frac{3}{5}}$.

Next,

   (d) For complex numbers $z$ and $w$, is there an upper bound on $|z^w|$ in terms of $|z|$ and $|w|$? Justify your answer.

4. For a complex number $z \neq 0$, show that $z^w$ has:

   (a) One value if $w$ is an integer.

   (b) Exactly $q$ different values if $w = p/q$ is a rational number, where $p$ and $q$ are integers, $q > 0$, and $\gcd(p, q) = 1$ (i.e, $p/q$ is in lowest terms).

   (c) Infinitely many different values if $w$ is not a rational number.

5. Consider, as mentioned in class, the problem of classifiying all linear transformations $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which preserve angles between vectors and have positive determinant.

(a) Give a geometric description of all such linear transformations and justify (i.e., prove) your statement.

Your proof will probably mostly involve geometric arguments (as in the argument in class for linear transformations which preserve length).

Using the standard basis $\{(1,0),(0,1)\}$ we can write any linear transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ as a matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$.

(b) The matrices for the linear transformations preserving angles (and with positive determinant) all have a particular form. I.e., there is a pattern on the entries $a$, $b$, $c$, and $d$ such that a linear transformation preserves angles and has positive determinant if and only if $a$, $b$, $c$, and $d$ follow this pattern. Find the pattern and prove your answer (both directions, since the question is if and only if).

NOTE: As part of your argument for (b) you should certainly be using your answer from part (a)!

REMARK: You may be wondering "what does the condition on positive determinant mean for a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$?" As a reminder, suppose that $v$ and $w$ are two linearly independent vectors. This means that $w$ cannot lie on the line spanned by $v$, and so to rotate from $v$ to $w$ there is a well-defined "closest direction", either clockwise or counter-clockwise.

If $T$ is a linear transformation $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with nonzero determinant, then $T(v)$ and $T(w)$ are also linearly independent. The sign of the determinant tells you what happens to the direction of rotation from one vector to another under the linear transformation. If the determinant is positive, the direction you rotate going from $T(v)$ to $T(w)$ will be the same (either clockwise or counterclockwise) as the direction you rotate going from $v$ to $w$. If the determinant of $T$ is negative, then the direction you rotate going from $T(v)$ to $T(w)$ will be the opposite of the direction you rotate going from $v$ to $w$.