Math 326 Homework Assignment 4

DUE DATE: Oct. 14, 2010

1. For each of the following harmonic functions $u(x, y)$, find a harmonic conjugate v.

(a) $u(x, y) = \sinh(x)\sin(y) + 6x$ (b) $u(x, y) = 4x^3y - 4xy^3 + 2xy$

2. Let $u(x, y) = \ln|\sqrt{x^2 + y^2}| = \frac{1}{2}\ln|x^2 + y^2|$. Note that the domain of definition of $u(x, y)$ is $S = \mathbb{C} \setminus \{0\}$.

(a) Is S simply connected?

(b) Verify that $u(x, y)$ is a harmonic function on S.

(c) Suppose we want to find a harmonic conjugate $v(x, y)$ for u. Write down the vector field $\vec{\mathbf{G}} = (G_1, G_2)$ so that "v is a harmonic conjugate of u" is the same as the condition that $\text{grad}(v) = \vec{\mathbf{G}}$.

(d) Sketch the vector field $\vec{\mathbf{G}}$. At a point (x_0, y_0) at distance r away from the origin, what is the length of $\vec{\mathbf{G}}(x_0, y_0)$?

(e) If v exists (and $\text{grad}(v) = \vec{\mathbf{G}}$) then $\vec{\mathbf{G}}$ tells us how v changes as we move in the xy-plane. Suppose we start at a point (x_0, y_0) at distance r away from the origin, and go once counterclockwise around the circle of radius r. What (according to the gradient) must happen to the value of v as we go around the circle?

(f) Can a harmonic conjugate v of u exist on all of S?

3. Let $\mathbf{F}$ be the function $\mathbf{F}(x, y) = (x^2 - y^2, 2xy)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(a) Find the tangent vectors to the lines $x = 1$ and $x + y = 2$ at $(1, 1)$ and show that they meet at angle $\frac{\pi}{4}$ (or $\frac{3\pi}{4}$, depending on how you orient the lines).

(b) Applying $\mathbf{F}$ turns the lines into curves. Find the tangent vectors to the curves at the point $(0, 2)$ $(= \mathbf{F}(1, 1))$ where the curves meet. (Do this by parametrizing the lines and plug the parameterizations into $\mathbf{F}$).

(c) Show that the new tangent vectors again meet at the same angle as in (a).

(d) Calculate the derivative matrix $\mathbf{DF}(1, 1)$, and show that the answer to (b) is obtained by applying the linear transformation $\mathbf{DF}(1, 1)$ to the vectors from (a).

(e) Show (perhaps using a previous homework question) that the matrix $\mathbf{DF}(1, 1)$ represents a linear transformation which preserves angles.

(f) Explain why conformal maps $\mathbf{F}\colon S \longrightarrow \mathbb{R}^2$ from an open set S are the same as complex differentiable maps $f\colon S \longrightarrow \mathbb{C}$ whose derivative is not zero at any point of S.

4.

(a) If $x, y \in \mathbb{R}$ show that $\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$, i.e., find the real and imaginary parts of $\sin(z)$.

(b) Use the answer to (a) to show that the statement "$|\sin(z)| \leq 1$ for all $z \in \mathbb{C}$" is *not* true. (This is one of the facts about sin and cos which do not remain true over the complex numbers, and as we will see later, such a bound is not true for any entire function.)

SUGGESTION: For part (a) it is probably easier to start on the right side of the equality, write out the definitions of sin, cos, sinh, cosh, and then do some algebra, with the goal of getting a function that depends only on $z = x + iy$.